LEARNING OBJECTIVES

After studying this unit you will be able to:

• Formulate the Linear Programming Problem
• Solve the Linear Programming Problem with two variables using graphical method
• Solve the Linear Programming Problem using simplex method
• Recognize problems that Linear Programming can handle

10.1 Introduction

Linear programming is a mathematical technique for determining the optimal allocation of resources and achieving the specified objective when there are alternative uses of the resources like money, manpower, materials, machines and other facilities. The objective in resource allocation may be either cost minimization or profit maximization. The technique of linear programming is applicable to all problems in which the total effectiveness about the use of resources can be expressed as a linear function of individual allocations, and the limitations on resources give rise to linear equalities or inequalities of individual allocations. The adjective linear, is to be particularly noted here. It implies that all the limitations or constraints and the objective must be expressed as linear functions.

Although the technique of linear programming is applicable to all such problems, there are some more easy methods for solving specific categories of problems. Special algorithms have been evolved to ease up computational task. The algorithms have acquired special names of their own. This has led to the following categories of the linear programming problems:

(i) General Linear Programming Problems.
(ii) Transportation Problems.
(iii) Assignment Problems.

We know that there is an opportunity cost for scarce resources that should be included in the relevant cost calculation for decision making and variance calculations. In practice several resources may be scarce. The opportunity cost of these scarce resources can be determined by the use of linear programming techniques. Our objective in this chapter is to examine linear programming techniques and to consider how they can be applied to some specific types of decisions that a firm may have to make.
10.2 Advanced Management Accounting

General Linear Programming Problems: An examination of the following simple example should illustrate the basic concepts of linear programming problem (abbreviated as LPP).

Illustration 1

A small scale industry unit manufactures two products, $X_1$ and $X_2$ which are processed in the machine shop and the assembly shop. The time (in hours) required for each product in the shops are given in the matrix below. Profit per unit is also given along.

<table>
<thead>
<tr>
<th>Product</th>
<th>Machine shop</th>
<th>Assembly shop</th>
<th>Profit per unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>2 hours</td>
<td>4 hours</td>
<td>₹ 3</td>
</tr>
<tr>
<td>$X_2$</td>
<td>3 hours</td>
<td>2 hours</td>
<td>₹ 4</td>
</tr>
<tr>
<td>Total time available (hours) (in a day)</td>
<td>16</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Assuring that there is unlimited demand for both products how many units of each should be produced every day to maximize total profit?

Solution

Let $x_1$ and $x_2$ be the number of units of $X_1$ and $X_2$ respectively that maximize the profit.

To begin solving the problem, let us restate the information in mathematical form. To do so, we introduce a term "objective function". This refers to the expression which shows the relationship of output to profit.

\[ 3x_1 = \text{total daily profit from the sale of } x_1 \text{ units product } X_1 \]
\[ 4x_2 = \text{total daily profit from the sale of } x_2 \text{ units product } X_2 \]

Objective function Max Profit = $3x_1 + 4x_2$

Department time constraints

Time used in making $x_1$ and $x_2$ units of two products must not exceed the total daily time available in two shops. In other words, (the hours required to make 1 unit of product $X_1$) multiplied by (the number of units of product $X_1$) plus (the hours required to make 1 unit of product $X_2$) multiplied by (the number of units of product $X_2$) must be less than or equal to the daily time available in each shop. Mathematically it is stated as

\[ 2x_1 + 3x_2 \leq 16 \quad \text{Machine shop constraint} \]
\[ 4x_1 + 2x_2 \leq 16 \quad \text{Assembles shop constraint} \]

In order to obtain meaningful answers, the values calculated for $x_1$ and $x_2$ cannot be negatives; they must represent real product $X_1$ and product $X_2$. Thus, all elements of the solution to the formulated linear programming problem must be either greater than or equal to 0 ($x_1 \geq 0, x_2 \geq 0$). The machine and assembly shops constraints are called structural constraints.

The problem can be summarized in a mathematical form as below:

Maximize:

\[ \text{Profit} = 3x_1 + 4x_2 \]
Subject to the constraints

\[ 2x_1 + 3x_2 \leq 16 \]
\[ 4x_1 + 2x_2 \leq 16 \]
\[ x_1 \geq 0, x_2 \geq 0. \]

\[
\text{Method of Linear Programming}
\]
\[
\text{Graphical Method} \quad \text{Simplex Method}
\]

### 10.2 Graphical Method

The graphical method discussed here provides necessary grounding in understanding the computational steps of the simplex algorithms, it itself being of little use in handling practical problems. It is possible to solve conveniently linear programming problems graphically as long as the number of variables (products, for example) is not more than two. Before we proceed to explain the graphical method, let us first know how linear inequalities are handled graphically.

**Linear inequalities in one variable and the solution space:** Any relation or function of degree one that involves an inequality sign is a linear inequality. It may be of one variable, or, may be of more than one variable. Simple examples of linear inequalities are those of one variable only; viz. \( x > 0, \ x < 0, \) etc.

![Graph showing solution space for \( x > 0 \) and \( x \leq 0 \)]

The values of the variables that satisfy an inequality are called the solution space, and is abbreviated as S.S. The solution spaces for (i) \( x > 0, \) (ii) \( x \leq 0 \) are shown by the dark portion in the above diagrams.

**Linear inequalities in two variables:** Now we turn to linear inequalities in two variables \( x, \) and \( y, \) and shade a few S.S.

![Graphs showing S.S. for linear inequalities in two variables]

Let us now consider a linear inequality in two variables given by \( 3x + y < 6. \)

The inequality mentioned above is true for certain pairs of numbers, \( (x, y) \) that satisfy \( 3x + y < 6. \) By trial, we may arbitrarily find such a pair to be \( (1,1) \) because \( 3 \times 1 + 1 = 4, \) and \( 4 < 6. \)
Linear inequalities in two variables may be solved easily by extending our knowledge of straight lines. For this purpose, we replace the inequality by an equality and seek the pairs of number that satisfy $3x + y = 6$. We may write $3x + y = 6$ as $y = 6 - 3x$, and draw the graph of this linear function.

Let $x = 0$ so that $y = 6$. Let $y = 0$, so that $x = 2$

Any pair of numbers $(x,y)$ that satisfies the equation $y = 6 - 3x$ falls on the line AB. Therefore, if $y$ is to be less than $6 - 3x$ for the same value of $x$, it must assume a value that is less than the ordinate of length $6 - 3x$.

All such points $(x,y)$ for which the ordinate is less than $6 - 3x$ lie below the line AB.
The region where these points fall is indicated by an arrow and is shaded too in the diagram above.

Now we consider two inequalities

\[ 3x + y < 6 \text{ and } x - y < -2 \]

being satisfied simultaneously by \( x \) and \( y \).

The pairs of numbers \((x, y)\) that satisfy both the inequalities may be found by drawing the graphs of the two lines \( y = 6 - 3x \) and \( y = 2 + x \), and determining the region where both the inequalities hold. It is convenient to express each equality with \( y \) on the left-side and the remaining terms in the right-side.

The first inequality \( 3x + y < 6 \) is equivalent to

\[ y < 6 - 3x \]

and requires the value of \( y \) for each \( x \) to be less than that of \( y = 6 - 3x \). This inequality is therefore satisfied by all points lying below the line \( y = 6 - 3x \). The region where these points fall has been shaded in the previous diagram.

We consider the second inequality \( x - y < -2 \), and note that this is equivalent to \( y > 2 + x \).

It requires the value of \( y \) for each \( x \) to be larger than that of \( y = 2 + x \). This inequality is, therefore, satisfied by all points lying above the line \( y = 2 + x \).

The region is indicated by an arrow and is shaded too in the diagram given below:

Let \( x = 0 \) so that \( y = 2 \); Let \( y = 0 \) so that \( x = -2 \).

By superimposing the above two graphs we determine the common region ACD in which the pairs \((x,y)\) satisfy both the inequalities.
Note: [1] The inequalities \(3x + y \leq 6\) and \(x - y \leq 2\) differ from the preceding ones in that these also include equality signs. It means that the points lying on the corresponding lines are also included in the region.

[2] The procedure may be extended to any number of inequalities.

**Graphical solution for linear programming problem:** We consider the problem of maximizing a linear function \(Z = x + 2y\) subject to the restrictions

\[
x \geq 0, \ y \geq 0, \ x \leq 6, \ y \leq 7, \ x + y \leq 12
\]

We note that the above inequalities may be grouped as follows:

\[
\begin{align*}
\{x \geq 0\} & \quad \{y \geq 0\} \\
\{x \leq 6\} & \quad \{y \leq 7\} \\
\{x + y \leq 12\}
\end{align*}
\]
By superimposing the above three diagrams, we determine the common region in the $xy$ plane where all the five inequalities are simultaneously satisfied.
This common region is known as feasible region or the solution set (or the polygonal convex sets).

A region is said to be bounded if it can be totally included within a (very large) circle. The shaded region enclosed by deep lines in the previous diagram is bounded, since it can be included within a circle.

The objective function attains a maximum or a minimum value at one of the corner points of the feasible solution known as extreme points of the solution set. Once these extreme points (the points of intersection of lines bounding the region) are known, a compact matrix representation of these points is possible. We shall denote the matrix of the extreme points by E.

The coefficients of the objective function may also be represented by a column vector. We shall represent this column vector by C.

The elements in the product matrix EC shows different values, which the objective function attains at the various extreme points. The largest and the smallest elements in matrix EC are respectively the maximum and the minimum values of the objective function. The row in matrix EC in which this happens is noted and the elements in that row indicate the appropriate pairing and are known as the optimal solution.

In the context of the problem under consideration.

\[
\begin{bmatrix}
0 & 0 \\
5 & 7 \\
6 & 0 \\
6 & 6
\end{bmatrix}
\begin{bmatrix}
1 \\
2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
5 \\
1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
7 & 1 & 7 & 2 \\
1 & 0 & 2 & 6 \\
1 & 6 & 6 & 2
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix}
\]

The given objective function viz. \(Z = x + 2y\) is maximum at the points (5, 7) present in the third row of the matrix E. Thus the optimal solution is \(x = 5, y = 7\), and the maximum value of the objective function is 19.

We now list the steps to be followed under graphical solution to a linear programming problem.

Step 1. Determine the region that satisfies the set of given inequalities.

Step 2. Ensure that the region is bounded*. If the region is not bounded, either there are

* It is inconceivable for a practical problem to have an unbounded solution.
additional hidden conditions which can be used to bound the region or there is no solution to the problem.

**Step 3.** Construct the matrix $E$ of the extreme points, and the column vector $C$ of the objective function.

**Step 4.** Find the matrix product $EC$. For maximization, determine the row in $EC$ where the largest element appears; while for minimization, determine the row in $EC$ where the smallest element appears.

**Step 5.** The objective function is optimized corresponding to the same row elements of the extreme point matrix $E$.

**Note:** If the slope of the objective function be same as that of one side of feasible region, there are multiple solutions to the problem. However, the optimized value of the objective function remains the same.

**Illustration 2**

A manufacturer produces two products $A$ and $B$, and has his machines in operation for 24 hours a day. Production of $A$ requires 2 hours of processing in machine $M_1$ and 6 hours in machine $M_2$. Production of $B$ requires 6 hours of processing in machine $M_1$ and 2 hours in machine $M_2$. The manufacturer earns a profit of ₹5 on each unit of $A$ and ₹2 on each unit of $B$. How many units of each product should be produced in a day in order to achieve maximum profit?

**Solution**

Let $x_1$ be the number of units of type $A$ product to be produced, and $x_2$ is that of type $B$ product to be produced. The formulation of the L.P.P. in this case is as below:

Maximize $Z = 5x_1 + 2x_2$

subject to the constraints.

\[
\begin{align*}
2x_1 + 6x_2 &< 24 \\
6x_1 + 2x_2 &< 24 \\
x_1 &\geq 0, \quad x_2 &\geq 0
\end{align*}
\]

For the line $2x_1 + 6x_2 = 24$

| let $x_1 = 0$, so that $x_2 = 4$ |
| let $x_2 = 0$, so that $x_1 = 12$ |

For the line $6x_1 + 2x_2 = 24$

| let $x_1 = 0$, so that $x_2 = 12$ |
| let $x_2 = 0$, so that $x_1 = 4$ |
The shaded portion in the diagram is the feasible region and the matrix of the extreme points $E_1, E_2, E_3$ and $E_4$ is

\[
\begin{bmatrix}
0 & 0 & E_1 \\
0 & 4 & E_2 \\
3 & 3 & E_3 \\
4 & 0 & E_4
\end{bmatrix}
\]

The column vector for the objective function is $C = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

The column vector the values of the objective function is given by

\[
EC = \begin{bmatrix}
0 & 0 \\
0 & 4 \\
3 & 3 \\
4 & 0
\end{bmatrix} \begin{bmatrix} 5 \\
5 \\
2 \\
2
\end{bmatrix} = \begin{bmatrix} 0 \times 5 + 0 \times 2 \\
0 \times 5 + 4 \times 2 \\
3 \times 5 + 3 \times 2 \\
4 \times 5 + 0 \times 2
\end{bmatrix} = \begin{bmatrix} 0 \\
8 \\
21 \\
20
\end{bmatrix}
\]

Since 21 is the largest element in matrix $EC$, therefore the maximum value is reached at the extreme point $E_3$ whose coordinates are $(3,3)$.

Thus, to achieve maximum profit the manufacturer should produce 3 units each of both the products A and B.
Summary of Graphical Method

It involves:

(i) Formulating the linear programming problem, i.e. expressing the objective function and constraints in the standardised format.

(ii) Plotting the capacity constraints on the graph paper. For this purpose normally two terminal points are required. This is done by presuming simultaneously that one of the constraints is zero. When constraints concerns only one factor, then line will have only one origin point and it will run parallel to the other axis.

(iii) Identifying feasible region and coordinates of corner points. Mostly it is done by breading the graph, but a point can be identified by solving simultaneous equation relating to two lines which intersect to form a point on graph.

(iv) Testing the corner point which gives maximum profit. For this purpose the coordinates relating to the corner point should put in objectives function and the optimal point should be ascertained.

(v) For decision – making purpose, sometimes, it is required to know whether optimal point leaves some resources unutilized. For this purpose value of coordinates at the optimal point should be put with constraint to find out which constraints are not fully utilized.

10.3 Trial & Error Method of solving Linear Programming Problem

Method of Solving LPP: Trial & Error Method (or Algebraic Approach): Graphical method cannot be used when there are more than 2 variables in a LPP., in such a case we use the simplex method which is highly efficient and versatile as also amenable to further mathematical treatment and offers interesting economic interpretations. However, its underlying concepts are rather lengthy to discuss and the student should patiently go through the following material on the trial and error method to gain a good grasp over the simplex technique.

Slack Variable

Consider the following example:

Example: Maximize \[ Z = 3x_1 + 4x_2 \]
Subject to \[ 2x_1 + 3x_2 \leq 16 \]
\[ 4x_1 + 2x_2 \leq 16 \]
\[ x_1, x_2 \geq 0 \]

The \( \leq \) type inequalities can be transformed into equalities by the addition of non negative variables, say \( s_1 \) and \( s_2 \) (known as slack variables) as below. The slack variables represent idle resources, therefore, they are to be positive or non negative. Further the contribution per unit of a slack variable is always taken as zero in the objective function of a LPP. This is so because profits are not made on unused resources.
And the objective function may be rewritten as below.

Maximize \( Z = 3x_1 + 4x_2 + 0s_1 + 0s_2 \)

**Surplus Variable**

In ≥ type inequalities, we subtract a variable (called the **surplus variable**) to make it an equality. The value of this variable can be interpreted as the excess amount of the resources utilized over and above the given level. The contribution per unit of a surplus variable is also taken as zero in the objective function.

**The Linear Programming Theorems:** The trial and error and simplex methods are based on the concept of slack variables and theorems described below:

**Extreme Point Theorem:** It states that an optimal solution to a LPP occurs at one of the vertices of the feasible region. This should be obvious from the discussion on the graphical method.

Now the vertices are defined by the intersection of equations. The first step of the method is, therefore, to convert the inequalities into equalities by the addition (or subtraction) of the slack (or surplus variables) depending on the direction of the inequality.

It is to be noted that the system of equations (A) above has more variables than the number of equations. Such a system of equations has an infinite number of solutions; yet it has finite and few vertices, the co-ordinates of which can be determined by applying the Basis theorem.

**Basis Theorem:** It states that for a system of \( m \) equations in \( n \) variables (where \( n > m \)) has a solution in which at least \((n-m)\) of the variables have value of zero as a vertex. This solution is called a basic solution.

Extreme point theorem can be extended to state that the objective function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible in that they have -ve co-ordinates and have to be dropped in view of the non-negativity condition on all variables including the slack and surplus variables.

Consider the following LPP for elucidating the basis theorem.

Maximize \( Z = 3x_1 + 4x_2 \)

Subject to \( 2x_1 + 3x_2 = 16 \)
\( 4x_1 + 2x_2 = 16 \)
\( x_1, x_2 \geq 0 \).

Introducing slack variables \( s_1 \) and \( s_2 \), we have

Maximize \( Z = 3x_1 + 4x_2 + 0s_1 + 0s_2 \)

Subject to \( 2x_1 + 3x_2 + 1s_1 + 0s_2 = 16 \)
\( 4x_1 + 2x_2 + 0s_1 + 1s_2 = 16 \)
\( x_1, x_2, s_1, s_2 \geq 0 \) \( \ldots (B) \)
Here \( n \) (number of variables) = 4 and \( m \) (number of equations) = 2. Thus \( n - m = 2 \). According to the Basis theorem we set \( 2 \) (= \( n - m \)) variables in (B) equal to zero at a time, solve the resulting system of equations and obtain a basic solution. Thus if we zeroise \( x_1 \) and \( x_2 \), the resulting system of equations would be

\[
\begin{align*}
1s_1 + 0s_2 &= 16 \\
0s_1 + 1s_2 &= 16 \\
\end{align*}
\]

......(C) Set 1 \((x_1 = x_2 = 0)\)

These equations directly yield \( s_1 = 16 \) and \( s_2 = 16 \) as the basic solution i.e. the co-ordinate of a vertex.

The other sets of equations, upon zeroising two variables at a time in (B), would be as follows:

\[
\begin{align*}
2x_1 + 3x_2 &= 16 \\
4x_1 + 2x_2 &= 16 \\
\end{align*}
\]

Set 2 \((s_1 = s_2 = 0)\)

\[
\begin{align*}
2x_1 + 0s_2 &= 16 \\
4x_1 + 1s_2 &= 16 \\
\end{align*}
\]

Set 3 \((x_2 = s_1 = 0)\)

\[
\begin{align*}
2x_1 + 1s_1 &= 16 \\
4x_1 + 0s_1 &= 16 \\
\end{align*}
\]

Set 4 \((x_1 = s_2 = 0)\)

\[
\begin{align*}
3x_2 + 1s_1 &= 16 \\
2x_2 + 0s_1 &= 16 \\
\end{align*}
\]

Set 5 \((x_1 = s_2 = 0)\)

\[
\begin{align*}
3x_2 + 0s_2 &= 16 \\
2x_2 + 1s_2 &= 16 \\
\end{align*}
\]

Set 6 \((x_1 = s_1 = 0)\)

It is a simple matter to solve these six sets of simultaneous equations and obtain the six basic solutions i.e. co-ordinates of the six vertices of the feasible region. The solutions are given below. The student may verify these by solving each of the sets of simultaneous equations.

<table>
<thead>
<tr>
<th>Set</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s_1 = 16, s_2 = 16 )</td>
</tr>
<tr>
<td>2</td>
<td>( x_1 = 2, x_2 = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>( x_1 = 8, s_2 = -16 )</td>
</tr>
<tr>
<td>4</td>
<td>( x_1 = 4, s_1 = 8 )</td>
</tr>
<tr>
<td>5</td>
<td>( x_2 = 8, s_1 = -8 )</td>
</tr>
<tr>
<td>6</td>
<td>( x_2 = 16/3, s_2 = 16/3 )</td>
</tr>
</tbody>
</table>

Since solutions of set 3 and 5 yield a negative co-ordinate each, contradicting thereby the non-negativity of constraints, thus these are infeasible and are to be dropped from consideration.
Now according to the basis theorem the optimal solution lies at one of these vertices. By substituting these co-ordinates the values of objective function are derived below:

<table>
<thead>
<tr>
<th>Set</th>
<th>Solution</th>
<th>Z (Profit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s_1 = 16$, $s_2 = 16$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>$x_1 = 2$, $x_2 = 4$</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>Infeasible</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$x_1 = 4$, $s_1 = 8$</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>Infeasible</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_2 = 16/3$, $s_2 = 16/3$</td>
<td>$21\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Thus solution of set 2 is optimal with a profit of 22. In this way we can solve a LPP simply by employing the theorems stated above; but the simplex method is a further improvement over the trial and error method. Following three are the inefficiencies in the trial and error method.

**Inefficiencies of Trial and Error Method:**

(a) In a LPP where $m$ and $n$ are larger, solving of numerous sets of simultaneous equations would be extremely cumbersome and time-consuming.

(b) Scanning the profit table we notice that we jump from profit 0 to 22 to $21\frac{1}{3}$ i.e. there are ups and downs. The simplex method ensures that successive solutions yield progressively higher profits, culminating into the optimal one.

(c) Some of the sets yield infeasible solutions. There should be means to detect such sets and not to solve them at all.

### 10.4 The Simplex Method for Maximization Problems

The simplex method is a computational procedure - *an algorithm* - for solving linear programming problems. It is an iterative optimizing technique. In the simplex process, we must first find an initial basis solution (extreme point). We then proceed to an adjacent extreme point. We continue moving from point to point until we reach an optimal solution. For a maximization problem, the simplex method always moves in the direction of steepest ascent, thus ensuring that the value of the objective function improves with each solution.

**Simplex Algorithm**

*Maximize*

\[
Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n
\]

*Subject to the constraints*

\[
\begin{align*}
&k_{11}x_1 + k_{12}x_2 + \ldots + k_{1n}x_n \leq b_1 \\
&k_{21}x_1 + k_{22}x_2 + \ldots + k_{2n}x_n \leq b_2 \\
&k_{m1}x_1 + k_{m2}x_2 + \ldots + k_{mn}x_n \leq b_m \\
x_1, x_2, \ldots, x_n \geq 0
\end{align*}
\]
• \( x_1, x_2, \ldots, x_n \) are the ‘Decision Variables’.
• \( c_j \) (\( j = 1, 2, \ldots, n \)) in the objective function are called the ‘Profit or Cost coefficients’.
• \( b_i \) (\( i = 1, 2, \ldots, m \)) are called ‘Resources’.
• Constants \( k_{ij} \) (\( i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \)) are called ‘Structural Coefficients’.

• An inequality of the “\( \leq \)” type is changed into an equality:
  By the addition of a non-negative slack variable. By adding a suitable positive quantity \( s_i \)
  to the left hand side, the inequality constraint can be written as:
  \[ k_{11}x_1 + k_{12}x_2 + \ldots + k_{1n}x_n + s_i = b_i \quad (i = 1, 2, \ldots, m) \]

• An inequality of the “\( \geq \)” type is changed into an equality:
  By the subtraction of a non-negative surplus variable. By subtracting positive quantity \( s_i \)
  from the left hand side, the inequality constraint can be written as:
  \[ k_{11}x_1 + k_{12}x_2 + \ldots + k_{1n}x_n - s_i = b_i \quad (i = 1, 2, \ldots, m) \]

Solution Steps:

\[ Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0s_1 + 0s_2 + \ldots + 0s_m \] (Objective function)

Subject to the constraints

\[
\begin{align*}
  k_{11}x_1 + k_{12}x_2 + \ldots + k_{1n}x_n + s_1 &= b_1 \\
  k_{21}x_1 + k_{22}x_2 + \ldots + k_{2n}x_n + s_2 &= b_2 \\
  k_{m1}x_1 + k_{m2}x_2 + \ldots + k_{mn}x_n + s_m &= b_m \\
  x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m \geq 0
\end{align*}
\]

• Note that slack variables have been allocated zero coefficients in the objective function
  as these variables typically contribute nil amount to the value of objective function.

INITIAL SIMPLEX TABLEAU

<table>
<thead>
<tr>
<th>( \mathbf{C}_i ) (Contribution per Unit)</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
<th>\ldots</th>
<th>( c_n )</th>
<th>0</th>
<th>0</th>
<th>\ldots</th>
<th>0</th>
<th>Minimum Ratio ( X_B / k_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{C}_j )</td>
<td>Basic Variables (B)</td>
<td>Value of Basic Variables ( b=(X_B) )</td>
<td>Coefficient Matrix</td>
<td>Identity Matrix</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_{B_1} )</td>
<td>( s_1 )</td>
<td>( b_1 = x_{B_1} )</td>
<td>( k_{11} )</td>
<td>( k_{12} )</td>
<td>\ldots</td>
<td>( k_{1n} )</td>
<td>1</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>( C_{B_2} )</td>
<td>( s_2 )</td>
<td>( b_2 = x_{B_2} )</td>
<td>( k_{21} )</td>
<td>( k_{22} )</td>
<td>\ldots</td>
<td>( k_{2n} )</td>
<td>0</td>
<td>1</td>
<td>\ldots</td>
</tr>
<tr>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>\ldots</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td>( \cdots )</td>
<td></td>
</tr>
<tr>
<td>( C_{B_m} )</td>
<td>( s_m )</td>
<td>( b_m = x_{B_m} )</td>
<td>( k_{m1} )</td>
<td>( k_{m2} )</td>
<td>\ldots</td>
<td>( k_{mn} )</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
</tr>
<tr>
<td>( Z = \sum C_{B_i} X_i )</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>\ldots</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Net Contribution per unit \( (C_i - Z) \)

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In the first row ‘C<sub>j</sub>’ we write the coefficients of the variables in the objective function (c<sub>1</sub>, c<sub>2</sub>, c<sub>3</sub> …….,c<sub>n</sub>, 0,0……0). These value will remain unchanged in the subsequent tableaus.

In the first column labeled ‘C<sub>B</sub>’, we list the coefficients of the current basic variables in the objective function.

In the second column ‘Basic Variables’ we place the basic variables (s<sub>1</sub>, s<sub>2</sub>, s<sub>3</sub> …..s<sub>m</sub>)

‘The Coefficient Matrix’ (under x<sub>1</sub>,x<sub>2</sub>……x<sub>n</sub>) in the tableau represents the coefficients of the decision variables in the constraints set.

‘The Identity Matrix’ (under s<sub>1</sub>, s<sub>2</sub>…….s<sub>m</sub>) in the initial simplex tableau represents the coefficients of the slack variables in the constraint set.

To get Z<sub>j</sub> row under a column, we multiply the entries of that column by the corresponding entries of C<sub>B</sub> column and add the product.

The last row ‘C<sub>i</sub> – Z<sub>j</sub>’ called the ‘Index Row or Net Evaluation Row’.

If all the elements or entries in the C<sub>i</sub> – Z<sub>j</sub> row are Negative or Zero then the current solution is optimum.

If current solution is not optimum, it can be further enhanced by eliminating one basic variable and replacing it by some non-basic one. For this

We now decide the variable to enter the solution mix. Column with largest positive entry in the C<sub>i</sub> – Z<sub>j</sub> row is called ‘key Column’ (indicated by ↑). The non-basic variable at the top of the column is the entering variable that will replace a basic variable.

Next step is to determine ‘leaving variable’ to be replaced. This can be trace by dividing each number in the X<sub>B</sub> column by the corresponding number in key column. We compute ratio (b<sub>1</sub>/k<sub>1j</sub>, b<sub>2</sub>/k<sub>2j</sub> ….. b<sub>m</sub>/k<sub>mj</sub>). The row corresponding to the minimum of these ratios is key row (indicated by ← ). Corresponding variable in the key row known as the leaving (departing) variable.

‘Key Element’ is the number that lies at the junction of the key column and key row.

Constructing Second Tableau.

- New values for the key row are computed by simply dividing every element of key row by key element.
- New values of the elements in the remaining rows for the new table can be obtained by performing elementary row operation on all rows so that all elements except the key element in the key column are zero. For each row other than the key row, we use following formula.
  \[ \text{New row number} = \text{Number in old rows} - \left( \frac{\text{corresponding number in key row} \times \text{(Corresponding fixed ratio)}}{\text{Key number}} \right) \]

Compute Z<sub>i</sub> and C<sub>i</sub> – Z<sub>j</sub> rows. If all the numbers in C<sub>i</sub> – Z<sub>j</sub> rows are either negative or Zero, an optimum solution has been obtained. If any of the numbers in C<sub>i</sub> – Z<sub>j</sub> row is positive repeat the steps as explained above until an optimum solution has been obtained.
The process is illustrated with the help of the following example:

**Illustration 3**

Maximize \[ Z = 3x_1 + 4x_2 \]

Subject to \[ 2x_1 + 3x_2 \leq 16 \] (machining time)

\[ 4x_1 + 2x_2 \leq 16 \] (assembly time)

\[ x_1 \geq 0, \quad x_2 \geq 0 \]

**Solution**

**Step 1. Obtaining an initial solution:** Obtain an initial solution that satisfies all the constraints of the problem. The simplex process requires that all constraints be expressed as equations. Therefore we must convert all inequalities into equalities.

Consider the constraint \[ 2x_1 + 3x_2 \leq 16 \]

Value on the left hand side of the inequality represents the amount of machine time a particular solution uses, while the quantity on right side of inequality sign represents the total amount of machine time available. Let \( s_1 \) be a variable which represents the unused machine time in this solution so that

\[ 2x_1 + 3x_2 + s_1 = 16 \]

Similarly, let \( s_2 \) represent the amount of assembly time that is available but not used so that

\[ 4x_1 + 2x_2 + s_2 = 16 \]

The variables \( s_1 \) and \( s_2 \) are referred to as slack variables.

Thus slack variables represent the quantity of a resource not used by a particular solution, and they are necessary to convert the constraint inequalities to equalities.

As we proceed with the simplex method, it will be helpful to place certain information in a table known as a simplex tableau.

The simplex solution is as follows.

**SIMPLEX TABLEAU-I**

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables (B)</th>
<th>Profit per Unit (C_B)</th>
<th>Qty [Value of Basic Variables b=X_b]</th>
<th>Coefficient Matrix</th>
<th>Identity Matrix</th>
<th>Replacement Ratio*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s_1</td>
<td>0</td>
<td>16</td>
<td>( x_1 ) \</td>
<td>( x_2 ) \</td>
<td>( s_1 )</td>
</tr>
<tr>
<td>2/3</td>
<td>s_2</td>
<td>0</td>
<td>16</td>
<td>( x_1 ) \</td>
<td>( x_2 ) \</td>
<td>( s_1 )</td>
</tr>
</tbody>
</table>

\[ Z_j = \sum C_B x_i \]

Net Contribution per unit \((C_j - Z_j)\)

<table>
<thead>
<tr>
<th>3</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>Replacement Ratio*</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>16/3</td>
</tr>
</tbody>
</table>

*Minimum ratio
SIMPLEX TABLEAU-II

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables (B)</th>
<th>Profit per Unit (C_B)</th>
<th>Qty [Value of Basic Variables b(=X_B)]</th>
<th>Coefficient Matrix</th>
<th>Identity Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x_1</td>
<td>x_2</td>
</tr>
<tr>
<td>1/4</td>
<td>x_2</td>
<td>4</td>
<td>16/3</td>
<td>2/3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>s_2</td>
<td>0</td>
<td>16/3</td>
<td>8/3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8/3</td>
</tr>
</tbody>
</table>

\[ Z_j = \sum C_{Bj}X_j \]

Net Contribution per unit \( (C_j - Z_j) \)

The main simplex table, aside from headings, is seen divided into 3 horizontal sub-tables, I to III. There are 6 main headings 1 to 6 as explained below for sub-table I.

Col. 1. Fixed ratio—explanation deferred.

Col. 2. Program \([Basic Variables (B)]\)—it contains the basic variables that are included in the solution. Initially, in sub-table I, there are slack variables, this means we do not produce x_1 and x_2 products, thus the profit is zero.

Col. 3. In the column labeled 'Profit' are listed the objective co-efficients of the basic variables that are included in the specific program. Thus, the profit co-efficients of s_1 and s_2 are all zero \([C_B]\).

Col. 4. In the column labelled 'QTY' are listed the values of the basic variables b(=X_B) included in the solution. Since our initial program consists of 16 units of s_1, and 16 units of s_2, these values are listed in 'QTY' column. Any variable which is not listed under the 'Program' column is known as the non-basic variable, the value of each non-basic variable is zero.

The total profit contribution resulting from a specific program can be calculated by multiplying corresponding entries in the "profit per unit" column and the "quantity"
column and adding the product. Thus, total profit contribution in our first program is $16(0) + 16(0)$ i.e. zero.

Col. 5. This heading is divided in as many sub-headings as the number of real and slack variables. It indicates each of the variables with profit/unit as read from the objective function at its top. The body of the sub-table carries the co-efficients of the variables as read from the constraints. That the co-efficients of the basic variables constitute a unit matrix in this initial sub-table is to be noted. The numbers under the non basic variables represent substitution ratio. For example, the number 2 gives the rate of substitution between $x_1$, and $s_1$. If we wish to produce 1 unit of $x_1$, 2 unit $s_1$, must be sacrificed. The number 4 has similar interpretation. To produce 1 unit of $x_1$, 4 units of $s_2$ must be sacrificed. By using the same explanation, to produce 1 unit of $x_2$, we must sacrifice 3 units of $s_1$ and 2 units of $s_2$.

Col. 6. Replacement ratio as discussed earlier.

The numbers in the net evaluation row (NER) $C_j - Z_j$ give the net effect of exchange between one unit of each variable and basic variable. They are always zero under the basic variables. Under the non-basic variables, they can be positive, negative or zero. Thus numbers in NER, under each column represent the opportunity cost of not having one unit of the respective column variable in the solution, in other words, the number represent the potential improvement in the objective function that will result by introducing into the program one unit of the respective column variable.

Test for Optimality: In so far as the total profit contribution (obtained by multiplying corresponding entries in the 'Profit' column and 'QTY' column and adding the product) resulting from the initial program is zero, it can be improved and hence is not optimal program. In maximization problem, the program is optimal if each $C_j - Z_j$ is either zero or negative.

Computational aspects for the derivation of improved solutions are explained step by step below.

Step 1. Select the incoming variable in sub-table I in the NER. The NER entries in the first sub-table are simply the profit unit figures as read from the objective function or copied from headings 5. However there is a more rigorous method of making these entries which alone has to be followed in making entries in the subsequent NER’s and explanation on which is deferred. Selection of the incoming variable is simple indeed. The positive $(C_j - Z_j)$ in NER indicate the magnitude of opportunity cost of not including 1 unit of variables $x_1$ and $x_2$ respectively at this stage. We select the one with highest entry in NER as incoming variable and this happens to be $x_2$, with a value of 4. We note this selection by putting an arrow below 4 in the NER of sub-table I. The column with the arrow is known as the key column.

Step 2. Select the outgoing variable, for this we have to calculate how many units of $x_2$ can be brought in without exceeding the existing capacity of any one of the resources? Thus, we must calculate the maximum allowable number of units of $x_2$, that can be brought into the program.
without violating the non-negative constraints. For this, we compute the replacement ratio of col. 6. This is done by dividing col. 4 by the key column and we get 16/3 and 8 and as the ratio. The variable s₂, against the least ratio of 16/3 is selected as the outgoing variable and the fact is noted by putting an arrow against the least ratio of 16/3. This is maximum quantity of x₂ that can be produced at this stage without violating the non negative constraint. This row with the arrow is called the key row. The element at the intersection of the key row and the key column is known as the pivot or key element and it is encircled.

**Step 3.** Having noted the incoming variable (x₂) and the outgoing variable (s₁) we are ready to perform the row operations on sub-table-I and fill in columns 4 and 5 of sub-table II.

Before that however, we fill in cols. 2 and 3 of sub-table II which is straight forward; s₁ is replaced by x₂ in col. 2, s₂ stays in it and in col. 3 the profit/unit figure corresponding to x₂ is copied from heading while other figures remain the same.

**Transformation of key row:** The rule for transforming the key row is: Divide all the numbers in the key row by the key number. The resulting numbers form the corresponding row in the next table.

The key row of sub-table-I (under col. 4 and 5) is divided by the pivot element and this becomes the corresponding row of the sub-table-II. It reads 16/3, 2/3, 1, 1/3, 0

**Transformation of the non-key rows:** The rule for transforming a non-key row is: Subtract from the old row number (in each column) the product of the corresponding key-row number and the corresponding fixed ratio formed by dividing the old row number in the key column by the key number. The result will give the corresponding new row number.

This rule can be placed in the following equation form:

\[
\text{New row number} = \text{old row number} - (\text{corresponding number in key row} \times \text{corresponding fixed ratio})
\]

The key row in sub-table-I is multiplied by the fixed ratio of the non-key row. This leads to

\[
16 \times \frac{2}{3} = \frac{32}{3}, \quad 2 \times \frac{2}{3} = \frac{4}{3}, \quad 3 \times \frac{2}{3} = 2, \quad 1 \times \frac{2}{3} = \frac{2}{3}, \quad 0 \times \frac{2}{3} = 0,
\]

This, however, is a rough work. Entries are not made any where. The result of this multiplication are subtracted from the non-key row of sub-table-I to yield the non-key row of the sub-table-II as below:

\[
16 - \frac{32}{3} = \frac{16}{3}, \quad 4 - \frac{4}{3} = \frac{8}{3}, \quad 2 - 2 = 0, \quad 0 - \frac{2}{3} = -\frac{2}{3}, \quad 1 - 0 = 1
\]

These entries are made for the non-key row of sub-table-II.
Step 4. This consists of deriving the NER of sub-table-II. Each of its elements is derived by multiplying col. 3 of sub-table-II with col. 5, summing these up and subtracting the sum from the profit/unit is heading 5. Computations for each element in the 2nd NER are shown below:

\[
C_j - Z_j = \\
3 - \left( 4 \times \frac{2}{3} + 0 \times \frac{8}{3} \right) = \frac{1}{3} \\
4 - (4 \times 1 + 0 \times 0) = 0 \\
0 - \left( 4 \times \frac{1}{3} + 0 \times \frac{-2}{3} \right) = -\frac{4}{3} \\
0 - (4 \times 0 + 0 \times 1) = 0
\]

By applying these 4 steps on sub-table-II we derive sub-table-III. Since its NER does not have any positive element i.e. opportunity cost, we stop here. This is an indication of the optimal solution with \( x_1 = 2 \) and \( x_2 = 4 \) (and \( Z = 22 \)), for if we derive another sub-table by bringing in \( x_3 \) or \( x_4 \) into the solution we shall decrease the profit. To sum up, when, in a maximization problem, all the elements in an NER are negative or zero we stop further iterations. The student may himself derive sub-table-III as an exercise.

It is obvious that the student needs a plenty of practice to be thorough with the computationally tedious though otherwise very elegant and versatile simplex method. As a tip, the fractions in the simplex tables must be retained and by trying to decimalise them the student may land himself in trouble. Numerous examples and exercises follow to give understanding and practice to the students.

### 10.5 Simplex Method for Minimization Problems

The simplex algorithm applies to both maximization and minimization problems. The only difference in the algorithm involves the selection of the incoming variable. In the maximization problem it is the one with highest +ve NER element. Conversely, it is the most –ve variable that is selected as the incoming variable in a minimization problem. And if all elements in the NER are either positive or zero, it is the indication for the optimal solution and we stop there.

Let us take an example straightway to explain the solution procedure.

**Illustration 4**

A small township of 15,000 people requires, on the average, 300,000 gallons of water daily. The city is supplied water from a central water-works where the water is purified by such conventional methods as filtration and chlorination. In addition, two different chemical compounds: (i) softening chemical and (ii) health chemical are needed for softening the water and for health purposes. The waterworks plans to purchase two popular brands that contain these chemicals. One unit of Chemico Corporation’s product gives 8 pounds of softening chemical and 3 pounds of health chemical. One unit of Indian Chemical’s product contains 4 pounds and 9 pounds per unit, respectively, for the same purposes.
To maintain the water at a minimum level of softness and to meet a minimum programme of health protection, experts have decided that 150 and 100 pounds of the two chemicals that make up each product must be added to water daily. At a cost of ₹8 and ₹10 per unit respectively for Chemico’s and Indian Chemical’s products, what is the optimal quantity of each product that should be used to meet the minimum level of softness and minimum health standard?

Solution
The relevant data may be tabulated as below:

<table>
<thead>
<tr>
<th>Chemical</th>
<th>Brand</th>
<th>Daily Requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chemico</td>
<td>Indian</td>
</tr>
<tr>
<td>(i) Softening</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>(ii) Health</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

Let us formulate the LPP as below

Minimize (cost), \[ Z = 8x + 10y \]

Subject to

\[ 8x + 4y \geq 150 \]
\[ 3x + 9y \geq 100 \]

and \[ x \geq 0, y \geq 0 \]

Physical Interpretation of Surplus Variables: Let \( s_1 \) and \( s_2 \) represent the extra units (if any) of the two chemicals over 150 and 100 units respectively. Their magnitude being such as to just convert the inequalities into equalities, thereby restricting them to non-negative value. The problem may then be restated as below:

Minimise (cost) \[ Z = 8x + 10y + 0s_1 + 0s_2 \]

Subject to

\[ 8x + 4y - s_1 = 150 \]
\[ 3x + 9y - s_2 = 100 \]

and \[ s_1 \geq 0, s_2 \geq 0, x \geq 0, y \geq 0. \]

Artificial Slack Variables: If \( x \) and \( y \) are set equal to zero \( s_1 \) and \( s_2 \) turn out to be negative, violating the non-negativity restriction. Therefore, to circumvent this, we introduce another similar device of artificial slack variables. Let us represent these by \( A_1 \) and \( A_2 \) respectively. The problem then becomes:
Minimize cost \( Z = 8x + 10y + 0s_1 + 0s_2 + MA_1 + MA_2 \)

Subject to
\[
\begin{align*}
8x + 4y - s_1 + A_1 &= 150 \\
3x + 9y - s_2 + A_2 &= 100
\end{align*}
\]
All variables \( \geq 0 \)

Physical Interpretation of the Artificial Variables.

These are imaginary brands, each unit containing 1 unit of the pertinent chemical. Both are restricted to non-negatives. Whereas surplus variables have zeros as their cost co-efficient, each artificial variable is assigned an infinitely large cost co-efficient (usually denoted by \( M \)).

Uses of artificial variables

1. It is our basic assumption that none of the basic variables in the L.P. problem can have a negative value. Thus, artificial variable is added to act as basic variable in a particular equation and hence it avoids the possibility of getting negative values for basic variables.

2. These artificial variables are such that their objective function co-efficients impose a huge and hence unacceptable penalty. In the case of maximization problem, the objective function is modified by subtracting a quantity \( MA_1 \) where \( M \) is arbitrarily large value and \( A_1 \) is the artificial variable. For minimization problem, the objective function is modified by adding the quantity \( MA_1 \). Thus artificial variables enable us to make a convenient and correct start in obtaining an initial solution.

3. The artificial variables with the high penalty once replaced by a real variable will never enter the optimal program. Hence the solution to the modified problem will give the optimal solution to the original problem.

Designing the Initial Program: It is obtained by letting each of the variables \( x, y, s_1 \) and \( s_2 \) assume value of zero, i.e., retaining artificial variables in the initial solution.

SIMPLEX TABLEAU-I

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables ((B))</th>
<th>( C_B )</th>
<th>Value of Basic Variables ( b(=X_B) )</th>
<th>( x )</th>
<th>( y )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>4/9</td>
<td>( A_1 ) M</td>
<td>150</td>
<td>8</td>
<td>4</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
<td>150/4</td>
</tr>
<tr>
<td></td>
<td>( A_2 ) M</td>
<td>100</td>
<td>3</td>
<td>9</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>←</td>
<td>100/9</td>
</tr>
</tbody>
</table>

\[
Z_j = \sum C_B X_B = 11M + 13M - M - M + M + M
\]

\[
C_j - Z_j = 8 - 11M (10 - 13M) M 0 0
\]

* Largest negative value
Initial Solution:

\[ \Delta = C_j - Z_j \]
\[ \begin{align*}
    x & : 8 - (8M + 3M) = 8 - 11M \\
    y & : 10 - (4M + 9M) = 10 - 13M \\
    s_1 & : 0 - (-M + 0) = M \\
    s_2 & : 0 - (0 - M) = M \\
    A_1 & : M - (M + 0) = 0 \\
    A_2 & : M - (0 + M) = 0 \\
\end{align*} \]

(A)

10–13M is the most – ve and, therefore, the column, under 'y' is the key column. Also, the ratio 100/9 is less than 150/4 (on extreme right). The pivot element, then, is 9. The outgoing variable is \(A_2\) being replaced by \(y\).

The key row is revised by dividing it through by 9, the pivot element and the results are as follows:

\[ \frac{100}{9}, \frac{3}{9}, 1, 0, \frac{-1}{9}, 0, \frac{1}{9} \]

The fixed ratio for revising the non-key row is 4/9. The revised figures for this row are computed below:

\[ \begin{align*}
    150 - 100 \times \frac{4}{9} &= 1350 - 400 = \frac{950}{9} \\
    8 - 3 \times \frac{4}{9} &= 72 - 12 = \frac{60}{9} \\
    4 - 9 \times \frac{4}{9} &= 0 \\
    -1 - 0 &= -1 \\
    0 + \frac{4}{9} &= \frac{4}{9} \\
    1 - 0 &= 1 \\
    0 - 1 \times \frac{4}{9} &= \frac{-4}{9} \\
\end{align*} \]

The new table is shown below:
### Simplex Tableau-II

<table>
<thead>
<tr>
<th>$C_j$</th>
<th>8</th>
<th>10</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Ratio</td>
<td>Basic Variables (B)</td>
<td>$C_B$</td>
<td>Value of Basic Variables $b(=X_B)$</td>
<td>$x$</td>
<td>$y$</td>
<td>$s_1$</td>
<td>$s_2$</td>
</tr>
<tr>
<td>3/60</td>
<td>$A_1$</td>
<td>M</td>
<td>950/9</td>
<td>60/9</td>
<td>0</td>
<td>-1</td>
<td>4/9</td>
</tr>
<tr>
<td></td>
<td>$y$</td>
<td>10/9</td>
<td>100/9</td>
<td>3/9</td>
<td>1</td>
<td>0</td>
<td>-1/9</td>
</tr>
</tbody>
</table>

$$Z_j = \sum C_B/X_j \times \begin{pmatrix} \frac{60}{9} M + \frac{30}{9} \\ \frac{4}{9} M - \frac{10}{9} \end{pmatrix}$$

For the non-key row the fixed ratio is $3/9, 60/9 = 3/60$. Revising the non-key row.

$\frac{950}{60}, 1, 0, -\frac{9}{60}, \frac{4}{60}, \frac{9}{60}, -\frac{4}{60}$

$\frac{100}{9} \times \begin{pmatrix} \frac{950}{9} \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = \frac{35}{6}$

$1 - \begin{pmatrix} 0 \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = 1$

$0 - \begin{pmatrix} -1 \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = \frac{1}{20}$

$\frac{-1}{9} \times \begin{pmatrix} \frac{4}{9} \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = \frac{-2}{15}$

$0 - \begin{pmatrix} -1 \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = \frac{-1}{20}$

$\frac{1}{9} \times \begin{pmatrix} -\frac{4}{9} \times \frac{3}{9} \times \frac{9}{60} \end{pmatrix} = \frac{2}{15}$
The result is tabulated below:

**SIMPLEX TABLEAU-III**

<table>
<thead>
<tr>
<th>Cj</th>
<th>8</th>
<th>10</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed Ratio</td>
<td>Basic Variables (B)</td>
<td>Cb</td>
<td>Value of Basic Variables b(=Xb)</td>
<td>x</td>
<td>y</td>
<td>s1</td>
<td>s2</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>x</td>
<td>8</td>
<td>95/6</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>y</td>
<td>10</td>
<td>35/6</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>Zj = ∑CjXj</td>
<td>8</td>
<td>10</td>
<td>-7</td>
<td>-4</td>
<td>7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Cj - Zj</td>
<td>7</td>
<td>10</td>
<td>4</td>
<td>0</td>
<td>M - 7/10</td>
<td>M - 4/5</td>
<td></td>
</tr>
</tbody>
</table>

Since all entries in Cj - Zj now are either zero or positive, the current solution is the optimum on and is given by:

\[
x = \frac{95}{6}, \quad y = \frac{35}{6}
\]

Hence the optimal cost is

\[
\frac{95}{6} \times 8 + \frac{35}{6} \times 10 = \frac{760 + 350}{6} = \frac{1,110}{6} = ₹185
\]

### 10.6 Marginal Value of a Resource

Let us consider illustration 3 under section 10.4 once again to explain this concept. In the said example there are two resources machining and assembly. The optimal tableau of that example is reproduced below. We see in its NER -5/4 and -1/8 below s1 and s2. The marginal value of the machining and assembly resources are then respectively 5/4 and 1/8. The implication of this is that if now (at the optimal stage) we wish to bring in s1 in the solution, the total profit will be reduced from 22 (the optimal) by 5/4 times the no. of units of s1 brought in the program. This is demonstrated below where we bring s1 in this program.

Optimal table of first illustration 3 under section 10.4 follows (rest omitted) s1 is brought in the next to optimal table.
Please note that marginal value of resource is synonymous with *opportunity cost or shadow price*.

### 10.7 Some Remarks

1. It may be desired to convert a maximization problem into a minimization one and *vice versa*. Mathematically, this can be accomplished by reversing signs though of just the objective function.

2. *Inequalities in the wrong direction: Consider the problem:*

   
   Maximize \( Z = x_1 + 5x_2 \)

   Subject to \( 3x_1 + 4x_2 \leq 6 \) \hspace{1cm} \ldots \quad (i) 

   \( x_1 + 3x_2 \geq 2 \) \hspace{1cm} \ldots \quad (ii) 

   \( x_1, x_2 \geq 0 \)

   (Whether to introduce slack or surplus or artificial variables depends on the type of inequality and has nothing to do with type of the problem i.e., maximization or minimization)

   The 2nd inequality is in the wrong direction. Upon introducing the "surplus" variable,

   \( x_1 + 3x_2 - s_2 = 2 \)

   If \( s_2 \) is taken in the initial solution it would be \(-ve\) when \( x_1 \) and \( x_2 \) are zero. To circumvent this, an artificial variable is also introduced in this inequality. The problem becomes:

   Maximize \( Z = x_1 + 5x_2 - MA_2 + 0s_1 + 0s_2 \)

   Subject to \( 3x_1 + 4x_2 + s_1 = 6 \)

   \( x_1 + 3x_2 - s_2 + A_2 = 2 \)

   \( x_1, x_2, s_1, s_2, A_2 \geq 0 \)

   * Profit has been reduced from 22 to 12 because 8 units of \( s_1 \) have been brought in the new solution, thereby the net reduction of profit 10 naturally equals \( 8 \times 5/4 \)
(Note that in maximization problems M always has –ve sign and in minimization problems M always has a +ve sign in the objective function).

The initial solution consists of $s_1$ and $A_2$. Several examples on inequalities in the wrong direction follow. Surplus variables can never come in initial solution.

3. Any linear programming problem can be re-formulated into what is known as its dual. Any of the primal (the original) or the dual may be selected for iterating by the simplex method. The selection is made on the basis of computational burden. Also the dual provides interesting insights into the methodology of the LP solution.

4. If two or more variables share the maximum positive co-efficient in the net evaluation row any one may be chosen for introduction for the new solution arbitrarily, viz., in $Z = 2x_1 + 2x_2 + x_3$ it matters little if $x_1$ or $x_2$ is chosen.

5. Lower bounds may be specified in an LPP. For example, over and above to the three usual constraints, it may be stipulated that $x_1$ cannot be less than 25 or 40 or $l_1$, i.e., $x_1 \geq l_1$. This can be handled quite easily by introducing a variable $y_1$ such that $x_1 = l_1 + y_1$. Substitute $x_1 = l_1 + y_1$ wherever it occurs and solve the LPP. Computations would be greatly reduced.

6. In all the simplex tables there is bound to be a unit matrix of size $p \times p$ where $p$ is the no. of rows (excluding net evaluation row). The columns that constitute such a unit matrix need not be adjacent.

7. In view of the tediousness of computational aspects it is useful to make a check at each iteration. This can be done by deriving the net evaluation row in two ways. (i) just like any other row in the simplex tableau by deriving its fixed ratio (ii) by summing the product of the quantities column with the profit/cost column and subtracting this sum from the original profit contribution or cost co-efficient of variable. These should tally. Also, having obtained the optimal solution it is desirable to verify it if it obeys the given constraints.

8. The simplex method, the graphical and trial and error methods, the dual approach provide several ways of doing an LPP. The student may want to do each LPP in more than one way for the sake of verification of the answer and practice.

10.8 Practical Applications of Linear Programming

In addition to its wide use in industrial and administrative applications, linear programming has extensive application to agricultural, aircraft and several military problems. These are briefly discussed below.

Industrial Applications: These are basically product-mix problems in which the general objective is to derive the optimal production and procurement plan for the time period under
consideration. The measure of effectiveness is either a defined return that is sought to be maximized or a defined cost that is to be minimized.

**Production Planning - Product Mix:** An industrial concern has available a certain production capacity on various manufacturing processes and has the opportunity to utilize this capacity to manufacture various products. Typically, different products will have different selling prices, will require different amounts of production capacity at the several processes, and therefore will have different unit profits; there may also be stipulations on maximum and/or minimum production levels. The problem is to determine the optimal mix so that the total profit is maximized.

**Blending Problems:** These problems are likely to arise when a product can be made from a variety of available raw materials of various compositions and prices. The manufacturing process involves blending (mixing) some of these materials in varying quantities to make a product conforming to given specifications. The supply of raw materials and specifications serve as constraints in obtaining the minimum cost material blend. The solution would state the number of units of each raw material which are to be blended to make one unit of product.

**Diet Problems:** Diet Problems are not much different from the blending problem.

**Trim Problems:** Trim Problems are applicable to paper industry where paper of standard width has to be cut into smaller width as per customer requirements with the objective of minimizing the waste produced.

**Advertising-Mix:** Advertising Mix problem is analogous to the product-mix problem.

**Financial Applications:** The investment portfolio selection problems can be satisfactorily handled by linear programming.

**Agriculture Applications:** These deal with somewhat different resource, but the objectives are nevertheless same - to maximize the return from the allocation activity or to minimize some defined cost.

**Flight Scheduling Applications:** Linear programming has been effectively applied to a variety of operational scheduling problems.

Other applications of linear programming include structural design, scheduling military tanker fleet, minimizing the number of carriers to meet a fixed schedule, the least ballast shipping required to meet a specific shipping program, cost cutting in business, fabrication scheduling, computations of maximum flows in network, steel production scheduling, stocks and flows the balancing of assembly lines, etc.

As such linear programming has contributed greatly to the enhancement of productivity at the firm, industry and national levels and continues to be promising in this regard.
LPP-Formulation

Illustration 5

WELL TYPE Manufacturing Company produces three types of typewriters; Manual type-writer, Electronic typewriters, and Deluxe Electronic typewriters. All the three models are required to be machined first and then assembled. The time required for the various models are as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>Machine Time (in hour)</th>
<th>Assembly Time (in hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manual Typewriter</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>Electronic Typewriter</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>Deluxe Electronic Typewriter</td>
<td>14</td>
<td>5</td>
</tr>
</tbody>
</table>

The total available machine time and assembly time are 3,000 hours and 1,200 hours respectively. The data regarding the selling price and variable costs for the three types are:

<table>
<thead>
<tr>
<th></th>
<th>Manual</th>
<th>Electronic</th>
<th>Deluxe Electronic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling Price (₹)</td>
<td>4,100</td>
<td>7,500</td>
<td>14,600</td>
</tr>
<tr>
<td>Labour, Material and other variable costs (₹)</td>
<td>2,500</td>
<td>4,500</td>
<td>9,000</td>
</tr>
</tbody>
</table>

The company sells all the three types on credit basis, but will collect the amounts on the first of next month. The labour, material and other variable expenses will have to be paid in cash. This company has taken a loan of ₹ 40,000 from a co-operative bank and this company will have to repay it to the bank on 1st April, 2003. The TNC bank from whom this company has borrowed ₹ 60,000 has expressed its approval to renew the loan.

The Balance Sheet of this Company as on 31.03.2012 is as follows:

<table>
<thead>
<tr>
<th>Liabilities</th>
<th>₹</th>
<th>Assets</th>
<th>₹</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity Share capital</td>
<td>1,50,000</td>
<td>Land</td>
<td>90,000</td>
</tr>
<tr>
<td>Capital Reserve</td>
<td>15,000</td>
<td>Building</td>
<td>70,000</td>
</tr>
<tr>
<td>General Reserve</td>
<td>1,10,000</td>
<td>Plant &amp; Machinery</td>
<td>1,00,000</td>
</tr>
<tr>
<td>Profit &amp; Loss a/c</td>
<td>25,000</td>
<td>Furniture &amp; Fixtures</td>
<td>15,000</td>
</tr>
<tr>
<td>Long term loan</td>
<td>1,00,000</td>
<td>Vehicles</td>
<td>30,000</td>
</tr>
<tr>
<td>Loan from TNC Bank</td>
<td>60,000</td>
<td>Inventory</td>
<td>5,000</td>
</tr>
<tr>
<td>Loan from Co-op. Bank</td>
<td>40,000</td>
<td>Receivables</td>
<td>50,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cash</td>
<td>1,40,000</td>
</tr>
<tr>
<td>Total</td>
<td>5,00,000</td>
<td>Total</td>
<td>5,00,000</td>
</tr>
</tbody>
</table>

The company will have to pay a sum of ₹ 10,000 towards the salary from top management executives and other fixed overheads for the month. Interest on long term loans is to be paid
every month at 24% per annum. Interest on loans from TNC and Co-operative Banks may be taken to be ₹ 1,200 for the month. Also this company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic typewriters to one of its valued customers next month.

Also make sure that the level of operations in this company is subject to the availability of cash next month. This company will also be able to sell all their types of typewriter in the market. The Senior Manager of this company desires to know as to how many units of each typewriter must be manufactured in the factory next month so as to maximize the profits of the company. Formulate this as a linear programming problem. The formulated problem need not be solved.

Solution

Let $X_1$, $X_2$ and $X_3$ denote the number of Manual, Electronic and Deluxe Electronic typewriters respectively to be manufactured in the factory next month. We are given the following data.

<table>
<thead>
<tr>
<th></th>
<th>Manual Typewriter</th>
<th>Electronic Typewriter</th>
<th>Deluxe Electronic Typewriter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling Price p.u. (₹)</td>
<td>4,100</td>
<td>7,500</td>
<td>14,600</td>
</tr>
<tr>
<td>Labour, Material and other variable costs p.u. (₹)</td>
<td>2,500</td>
<td>4,500</td>
<td>9,000</td>
</tr>
<tr>
<td>Profit contribution p.u. (₹)</td>
<td>1,600</td>
<td>3,000</td>
<td>5,600</td>
</tr>
</tbody>
</table>

The objective of the company is to maximize the profit hence the objective function is given by

Maximize $Z = 1,600 X_1 + 3,000 X_2 + 5,600 X_3 – (₹ 10,000 + ₹ 2,000 + ₹ 1,200)$

From the data given for time required for various models and the total number of hours available for machine time and assembly time, we get the following constraints:

$$15X_1 + 12 X_2 + 14X_3 < 3,000 \text{ (machine time restriction)}$$

and

$$4X_1 + 3X_2 + 5X_3 < 1,200 \text{ (assembly time restriction)}$$

The level of operations in the company is subject to the availability of cash next month. That is, the cash required for manufacturing various models should not exceed the cash available for the next month.

The cash requirements for $X_1$ units of Manual, $X_2$ units of Electronic and $X_3$ units of Deluxe Electronic typewriters are;

$$2,500 X_1 + 4,500 X_2 + 9,000 X_3 \quad \ldots \text{(i)}$$

*The cash availability for the next month from the balance sheet is as below:*

Cash availability (₹) = Cash balance (₹ 1,40,000) + Receivables (₹ 50,000)

− Loan to repay to co-operative bank (₹ 40,000)

− Interest on loan from TNC & co-operative banks (₹ 1,200)

− Interest on long term loans $\left( \frac{0.24 \times ₹ 1,00,000}{12} \right)$
10.32 Advanced Management Accounting

- (Top management salary and fixed overheads) (₹ 10,000)

Or Cash availability = ₹ 1,40,000 + ₹ 50,000 – (₹ 40,000 + ₹ 1,200 + ₹ 2,000 + ₹ 10,000)
= ₹ 1,90,000 – ₹ 53,200
= ₹ 1,36,800

From (i) & (ii), we get
2,500 X_1 + 4,500 X_2 + 9,000 X_3 ≤ ₹ 1,36,800

Further, the company has promised to deliver 2 Manual typewriters and 8 Deluxe Electronic Typewriters to one of its customers.

Hence, X_1 ≥ 2, X_2 ≥ 0, and X_3 ≥ 8

The linear programming formulation of the given problem is as follows:
Maximize Z = 1,600 X_1 + 3,000 X_2 + 5,600 X_3 – (₹ 10,000 + ₹ 2,000 + ₹ 1,200)

Subject to the constraints:
15X_1 + 12X_2 + 14X_3 ≤ 3,000
4X_1 + 3X_2 + 5X_3 ≤ 1,200
2,500X_1 + 4,500X_2 + 9,000X_3 ≤ 1,36,800
X_1 ≥ 2, X_2 ≥ 0, X_3 ≥ 8,

X_1, X_2, and X_3 can take only positive integral values.

Illustration 6

Consider a company that must produce two products over a production period of three months of duration. The company can pay for materials and labour from two sources: company funds and borrowed funds.

The firm faces three decisions:

1. How many units should it produce of Product 1?
2. How many units should it produce of Product 2?
3. How much money should it borrow to support the production of the two products?

In making these decisions, the firm wishes to maximize the profit contribution subject to the conditions stated below:

(i) Since the company’s products are enjoying a seller’s market, it can sell as many units as it can produce. The company would therefore like to produce as many units as possible subject to production capacity and financial constraints. The capacity constraints, together with cost and price data, are given in Table-1.
TABLE-1 Capacity, Price and cost data

<table>
<thead>
<tr>
<th>Product</th>
<th>Selling Price (₹ per unit)</th>
<th>Cost of Production (₹ per unit)</th>
<th>Required Hours per unit in Department</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>10</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>8</td>
<td>0.3</td>
</tr>
<tr>
<td>Available hours per production period of three months</td>
<td>500</td>
<td>400</td>
<td>200</td>
</tr>
</tbody>
</table>

(ii) The available company funds during the production period will be ₹ 3 lakhs.

(iii) A bank will give loans up to ₹ 2 lakhs per production period at an interest rate of 20 percent per annum provided the company's acid (quick) test ratio is at least 1 to 1 while the loan is outstanding. Take a simplified acid-test ratio given by

\[
\text{Surplus cash on hand after production + Accounts receivable} = \frac{\text{Bank Borrowing + Interest accrued thereon}}{1}
\]

(iv) Also make sure that the needed funds are made available for meeting the production costs.

Formulate the above as a Linear Programming Problem.

Solution

Let \( x_1 \) = No. of units of product 1 produced
\[ x_2 = \text{No of units of product 2 produced} \]
\[ x_3 = \text{Amount of money borrowed} \]

The profit contribution per unit of each product is given by the selling price minus the variable cost of production. Total profit may be computed by summing up the profit from producing the two products minus the cost associated with borrowed funds (if any):

The objective function is thus stated as

Maximize \( Z = (14-10) x_1 + (11-8) x_2 - 0.05 x_3 \)  \( (\text{Note that the interest rate is 20% per annum, hence 5% for a period of three months})\)

\[ = 4x_1 + 3x_2 - 0.05x_3 \]

Subject to the following constraints:

The production capacity constraints for each department as given by table 1 are:

\[ 0.5x_1 + 0.3x_2 \leq 500 \]  \( ...(1) \)
\[ 0.3x_1 + 0.4x_2 \leq 400 \]  \( ...(2) \)
\[ 0.2x_1 + 0.1x_2 \leq 200 \]  \( ...(3) \)

The funds available for production include both ₹ 3,00,000 cash that the firm possesses and any borrowed funds maximum up to ₹ 2,00,000. Consequently production is limited to the
extent that funds are available to pay for production costs. The constraint expressing this relationship is

Funds required for production \( \leq \) Funds available

i.e. \( 10x_1 + 8x_2 \leq 3,00,000 + x_3 \)

or \( 10x_1 + 8x_2 - x_3 \leq 3,00,000 \)  .....(4)

The borrowed funds constraint (from condition (iii) of the Question) is

\( x_3 \leq 2,00,000 \)  .....(5)

The constraint based on the acid – test condition is developed as follows:

\[
\frac{\text{Surplus cash on hand after production} + \text{Accounts receivable}}{\text{Bank Borrowing} + \text{Interest accrued thereon}} \geq 1
\]

or \( 3,00,000 + x_3 + 4x_1 + 3x_2 \geq (x_3 + 0.05x_3) \)

or – \( 4x_1 - 3x_2 + 0.05x_3 \leq 3,00,000 \)  .....(6)

Thus, the linear programming problem is given by

Maximize \( Z = 4x_1 + 0.3x_2 - 0.05x_3 \)

subject to

\[
0.5x_1 + 0.3x_2 \leq 500
\]

\[
0.3x_1 + 0.4x_2 \leq 400
\]

\[
0.2x_1 + 0.1x_2 \leq 200
\]

\[
10x_1 + 8x_2 - x_3 \leq 3,00,000
\]

\[
x_3 \leq 2,00,000
\]

\[
-4x_1 - 3x_2 + 0.05x_3 \leq 3,00,000
\]

Where \( x_1, x_2, x_3 \geq 0 \)

**Illustration 7**

North-East Aircraft company, which operates out of a central terminal has 8 aircraft of Type I, 15 aircraft of Type II, and 12 aircraft of Type III available for today’s flights. The tonnage capacities (in thousands of tons) are 4.5 for Type I, 7 for Type II and 4 for Type III.

The company dispatches its planes to cities A and B. Tonnage requirements in thousands of tons are 20 at city A and 30 at city B: excess tonnage capacity supplied to a city has no value. A plane can fly once only during the day.
The cost of sending a plane from the terminal to each city is given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>Type I</th>
<th>Type II</th>
<th>Type III</th>
</tr>
</thead>
<tbody>
<tr>
<td>City A</td>
<td>23</td>
<td>5</td>
<td>1.4</td>
</tr>
<tr>
<td>City B</td>
<td>58</td>
<td>10</td>
<td>3.8</td>
</tr>
</tbody>
</table>

Formulate the LPP model to minimize the air-transportation cost

**Solution**

Let \( x_{11} \) be the I plane coming to A, \( x_{12} \) the II planes coming to A; and so on.

<table>
<thead>
<tr>
<th>City</th>
<th>Type-I</th>
<th>Type-II</th>
<th>Type-III</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( x_{11} )</td>
<td>( x_{12} )</td>
<td>( x_{13} )</td>
</tr>
<tr>
<td>B</td>
<td>( x_{21} )</td>
<td>( x_{22} )</td>
<td>( x_{23} )</td>
</tr>
</tbody>
</table>

Minimize \( Z = 23x_{11} + 5x_{12} + 1.4x_{13} + 58x_{21} + 10x_{22} + 3.8x_{23} \)

Tonnage requirements of City A and B

\[ 4.5x_{11} + 7x_{12} + 4x_{13} = 20 \]
\[ 4.5x_{21} + 7x_{22} + 4x_{23} = 30 \]

Central Terminal has 8 aircraft of Type I, 15 aircraft of Type II, and 12 aircraft of Type III

\[ x_{21} + x_{11} \leq 8 \]
\[ x_{22} + x_{12} \leq 15 \]
\[ x_{23} + x_{13} \leq 12 \]
\[ x_{ij} \geq 0 \]

**Illustration 8**

The Voltex Company produces an air conditioner/heating unit. The company currently has firm orders for 6 months into the future. The company can schedule its production over the next 6 months to meet orders on either a regular or on overtime basis. Consider orders and the associated production costs for the next 6 months as follows:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Orders Cost/unit (₹)</td>
<td>590</td>
<td>610</td>
<td>650</td>
<td>700</td>
<td>500</td>
<td>700</td>
</tr>
<tr>
<td>Regular Production Cost/Unit (₹)</td>
<td>50</td>
<td>52</td>
<td>51</td>
<td>55</td>
<td>47</td>
<td>50</td>
</tr>
<tr>
<td>Overtime Production</td>
<td>62</td>
<td>58</td>
<td>63</td>
<td>60</td>
<td>55</td>
<td>52</td>
</tr>
</tbody>
</table>

Maximum no. of units which can be produced on regular and overtime basis are 500 and 300 respectively.

With 75 air conditioners in stock at the beginning of January, the company wishes to have at least 100 air conditioners in stock at the end of June. The inventory-carrying cost for air-conditioners is ₹10 per unit per month.
Solution
Let \( x_{ij} \) = number of units produced in month \( i \) \((i = 1, 2 \ldots, 6)\),
on a regular or overtime basis \((j = 1, 2)\)
\( y_i \) – number of units of ending inventory in month \( i \) \((i = 1, 2 \ldots, 6)\)

Minimize (cost)
\[
Z = 50x_{11} + 62x_{12} + 52x_{21} + 58x_{22} + 51x_{31} + 63x_{32} + 55x_{41} + 60x_{42} + 47x_{51} + 55x_{52} + 50x_{61} + 52x_{62} + 10(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)
\]
Subject to
\[
egin{align*}
75 + x_{11} + x_{12} - 590 &= y_1 \\
y_1 + x_{21} + x_{22} - 610 &= y_2 \\
y_2 + x_{31} + x_{32} - 650 &= y_3 \\
y_3 + x_{41} + x_{42} - 700 &= y_4 \\
y_4 + x_{51} + x_{52} - 500 &= y_5 \\
y_5 + x_{61} + x_{62} - 700 &= y_6 \\
y_6 &\geq 100 \text{ (Ending Inventory Constraints)}
\end{align*}
\]
\[
\begin{align*}
x_{11} &\leq 500; x_{12} \leq 300 \\
x_{21} &\leq 500; x_{22} \leq 300 \\
x_{31} &\leq 500; x_{32} \leq 300 \\
x_{41} &\leq 500; x_{42} \leq 300 \\
x_{51} &\leq 500; x_{52} \leq 300 \\
x_{61} &\leq 500; x_{62} \leq 300 \\
x_{ij} &\geq 0, \text{ all } i \text{ and } j \\
y_i &\geq 0, \text{ all } i
\end{align*}
\]

Illustration 9
A refinery makes 3 grades of petrol (A, B, C) from 3 crude oils (d, e, f). Crude can be used in any grade but the others satisfy the following specifications.

<table>
<thead>
<tr>
<th>Grade</th>
<th>Specifications</th>
<th>Selling Price per litre</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Not less than 50% crude d</td>
<td>8.0</td>
</tr>
<tr>
<td></td>
<td>Not more than 25% crude e</td>
<td></td>
</tr>
</tbody>
</table>
There are capacity limitations on the amount of the three crude elements that can be used:

<table>
<thead>
<tr>
<th>Crude</th>
<th>Capacity (000)</th>
<th>Price per litre (₹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>500</td>
<td>9.5</td>
</tr>
<tr>
<td>e</td>
<td>500</td>
<td>5.5</td>
</tr>
<tr>
<td>f</td>
<td>300</td>
<td>6.5</td>
</tr>
</tbody>
</table>

It is required to produce the maximum profit.

Formulation

Let there be

- $x_1$ litres of $d$ in $A$
- $x_2$ litres of $e$ in $A$
- $x_3$ litres of $f$ in $A$
- $y_1$ litres of $d$ in $B$
- $y_2$ litres of $e$ in $B$
- $y_3$ litres of $f$ in $B$
- $z_1$ litres of $d$ in $C$
- $z_2$ litres of $e$ in $C$
- $z_3$ litres of $f$ in $C$

Then

$$\frac{x_1}{x_1 + x_2 + x_3} \geq \frac{1}{2} \text{ i.e. } -x_1 + x_2 + x_3 \leq 0$$

$$\frac{x_2}{x_1 + x_2 + x_3} \leq \frac{1}{4} \text{ i.e. } -x_1 + 3x_2 - x_3 \leq 0$$

$$\frac{y_1}{y_1 + y_2 + y_3} \geq \frac{1}{4} \text{ i.e. } -3y_1 + y_2 + y_3 \leq 0$$

$$\frac{y_2}{y_1 + y_2 + y_3} \leq \frac{1}{2} \text{ i.e. } -y_1 + y_2 - y_3 \leq 0$$

Also

- $x_1 + y_1 + z_1 \leq 500,000$
- $x_2 + y_2 + z_2 \leq 500,000$
- $x_3 + y_3 + z_3 \leq 300,000$
10.38 Advanced Management Accounting

Sales Price
\[
\left[ (x_1 + x_2 + x_3) \times 8 + (y_1 + y_2 + y_3) \times 6.5 + (z_1 + z_2 + z_3) \right]
\]

Cost
\[
- \left[ (x_1 + y_1 + z_1) \times 9.5 \right]
\]
\[
- \left[ (x_2 + y_2 + z_2) \times 5.5 \right]
\]
\[
- \left[ (x_3 + y_3 + z_3) \times 6.5 \right]
\]

Profit = \(-1.5x_1 + 2.5x_2 + 1.5x_3 - 3.0y_1 + 1.0y_2 + 0.5y_3 - 4.0z_1 + 0.5z_2 - 1.0z_3\)

Max Z = \(-1.5x_1 + 2.5x_2 + 1.5x_3 - 3.0y_1 + 1.0y_2 + 0.5y_3 - 4.0z_1 + 0.5z_2 - 1.0z_3\)

Subject to
\[-x_1 + x_2 + x_3 \leq 0\]
\[-x_1 + 3x_2 - x_3 \leq 0\]
\[-3y_1 + y_2 + y_3 \leq 0\]
\[-y_1 + y_2 - y_3 \leq 0\]
\[x_1 + y_1 + z_1 \leq 500,000\]
\[x_2 + y_2 + z_2 \leq 500,000\]
\[x_3 + y_3 + z_3 \leq 300,000\]

Illustration 10

The vitamins V and W are found in two different foods, \(F_1\) and \(F_2\). The amount of vitamin in each of the two foods, respective prices per unit of each food, and the daily vitamin requirements are given in the following table. The data indicate that one unit of \(F_1\) contains 2 units of vitamin V and 3 units of vitamin W. Similarly one unit of \(F_2\) contains 4 units of vitamin V and 2 units vitamin W. Daily requirements of vitamin V is at least 40 units and vitamin W of at least 50 units.

The problem is to determine optimal quantities of foods \(F_1\) and \(F_2\) to be bought so that the daily vitamin requirements are met and, simultaneously the cost of buying the goods is minimized.

<table>
<thead>
<tr>
<th>Vitamin</th>
<th>Food</th>
<th>Daily Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(F_1)</td>
<td>(F_2)</td>
</tr>
<tr>
<td>V</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>W</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Cost/Unit of food (₹)</td>
<td>3</td>
<td>2.5</td>
</tr>
</tbody>
</table>
Formulation:
Let \( x_1 \) and \( x_2 \) be quantities of \( F_1 \) and \( F_2 \) respectively.

Minimize \( 3x_1 + 2.5x_2 \)
subject to \( 2x_1 + 4x_2 \geq 40 \)
\( 3x_1 + 2x_2 \geq 50 \)
\( x_1 \geq 0 \)
\( x_2 \geq 0 \)

Illustration 11
The Fine Paper Company produces rolls of paper used in cash registers. Each roll of paper is 500 ft. in length and can be produced in widths of 1, 2, 3 and 5 inch. The company's production process results in 500' rolls that are 12 inches in width. Thus the company must cut its 12 inch roll to the desired width. It has six basic cutting alternatives as follows:

<table>
<thead>
<tr>
<th>Cutting Alternative</th>
<th>( 1^* )</th>
<th>( 2^* )</th>
<th>( 3^* )</th>
<th>( 5^* )</th>
<th>Waste (Inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The maximum demand requirements for the four rolls are as follows:

<table>
<thead>
<tr>
<th>Roll Width (inches)</th>
<th>Demand Requirements (Rolls)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3,000</td>
</tr>
<tr>
<td>2</td>
<td>2,000</td>
</tr>
<tr>
<td>3</td>
<td>1,500</td>
</tr>
<tr>
<td>5</td>
<td>1,000</td>
</tr>
</tbody>
</table>

The company wishes to minimize the waste generated by its production meeting its demand requirements. Formulate the LP model.

Formulation. Let \( x_i \) be the number of times cutting alternative \( (i = 1, 2, \ldots, 6) \) is employed.

Minimise (waste produced) \( Z = x_3 + x_4 + x_5 + x_6 \)
subject to
\[
6x_1 + x_3 + 4x_6 \leq 3,000
\]
\[
3x_1 + 3x_2 + x_3 + 4x_5 + 2x_6 \leq 2,000
\]
\[
2x_2 + x_3 + 2x_4 + 1x_5 + 1x_6 \leq 1,500
\]
\[
1x_3 + 1x_4 \leq 1,000
\]
\[
x_j \geq 0, \text{ for all } j
\]
Illustration 12

The Agro Promotion Bank is trying to select investment portfolio for a cotton farmer. The bank has chosen a set of five investment alternatives, with subjective estimates of rates of return and risk, as follows:

<table>
<thead>
<tr>
<th>Investment</th>
<th>Annual Rate of Return</th>
<th>Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tax-free municipal bonds</td>
<td>6.0</td>
<td>1.3</td>
</tr>
<tr>
<td>Corporate bonds</td>
<td>8.0</td>
<td>1.5</td>
</tr>
<tr>
<td>High grade common stock</td>
<td>5.0</td>
<td>1.9</td>
</tr>
<tr>
<td>Mutual fund</td>
<td>7.0</td>
<td>1.7</td>
</tr>
<tr>
<td>Real estate</td>
<td>15.0</td>
<td>2.7</td>
</tr>
</tbody>
</table>

The bank officer in charge of the portfolio would like to maximize the average annual rate of return on the portfolio. However, the wealthy investor has specified that the average risk of the portfolio should not exceed 2.0; and does not want more than 20% of the investment to be put into real estate. Formulate an LP model for the problem.

Formulation

Let \( x_j \) be the percent of portfolio allocated to investment \( j \) (\( j = 1,2,3,4,5 \)).

Maximize (average annual rate of return on the portfolio)

\[
Z = 6x_1 + 8x_2 + 5x_3 + 7x_4 + 15x_5
\]

subject to (Average risk constraint)

\[
1.3x_1 + 1.5x_2 + 1.9x_3 + 1.7x_4 + 2.7x_5 \leq 2.0
\]

(Real estate constraint)

\[
x_5 \leq 0.2
\]

(Total investment constraint)

\[
x_1 + x_2 + x_3 + x_4 + x_5 = 1.0
\]

\[
x_1, x_2, x_3, x_4, x_5 > 0.
\]

Illustration 13

A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs ₹100 for preparation, requires 7 man-days of work and yields a profit of ₹30. An acre of wheat costs ₹120 for preparation, requires 10 man-days of work and yields a profit of ₹40. An acre of soyabeans costs ₹70 to prepare, requires 8 man days of work and yields a profit of ₹20. If the farmer has ₹100,000 for preparation and can count on 8000 man-days of work, how many acres should be allocated to each crop to maximize profits? Formulate an LP model.

Formulation: Let \( x_1, x_2 \) and \( x_3 \) designate the average of corn, wheat and soyabeans respectively.

Maximize

\[
Z = 30x_1 + 40x_2 + 20x_3
\]

subject to (Money)

\[
100x_1 + 120x_2 + 70x_3 \leq ₹100,000
\]

(Man-day)

\[
7x_1 + 10x_2 + 8x_3 \leq ₹8000
\]
Illustration 14

The managers of several cattle feed lots are interested in determining how many of each of several types of livestock feed to purchase in order to satisfy the nutritional requirements for their livestock. They wish to purchase these foods in a manner that minimizes the cost of feeding their livestock. Relevant costs and nutritional data is as below.

<table>
<thead>
<tr>
<th>Required Nutrient</th>
<th>Units of Nutritional Elements</th>
<th>Minimum Nutrient Requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Alfa</td>
<td>Corn</td>
</tr>
<tr>
<td>Nutrient A</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>Nutrient B</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Nutrient C</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>Cost per unit</td>
<td>₹1.00</td>
<td>1.25</td>
</tr>
</tbody>
</table>

**Formulation**

Let \( x_j \) to be number of units of food type \( j = (1,2,3,4) \) used

Minimize (Cost) = \( 1.00x_1 + 1.25x_2 + 0.95x_3 + 1.35x_4 \)

Subject to

(Nutrient A) \( 40x_1 + 50x_2 + 30x_3 + 60x_4 \geq 500 \)

(Nutrient B) \( 30x_1 + 60x_2 + 35x_3 + 40x_4 \geq 750 \)

(Nutrient C) \( 25x_1 + 30x_2 + 25x_3 + 50x_4 \geq 600 \)

\( x_1, x_2, x_3, x_4 \geq 0 \)

Illustration 15

The owner of Fancy Goods Shop is interested to determine, how many advertisements to release in the selected three magazines A, B and C. His main purpose is to advertise in such a way that exposure to principal buyers of his goods is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements released multiplied by the number of principal buyers. The following data are available:

<table>
<thead>
<tr>
<th>Particulars</th>
<th>Magazines</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>Readers</td>
<td>1.0 lakh</td>
</tr>
<tr>
<td>Principal Buyers</td>
<td>20%</td>
</tr>
<tr>
<td>Cost per Advertisement (₹)</td>
<td>8,000</td>
</tr>
</tbody>
</table>

The budgeted amount is at the most 1.0 lakh for the advertisement. The owner has already decided that magazine A should have no more than 15 advertisements and that B and C each gets at least 8 advertisements. Formulate a Linear Programming Model for this problem.
Solution

Let \( x_1, x_2 \) and \( x_3 \) denote the number of advertisements to be released in three magazines A, B and C respectively. Let \( Z \) denote the total exposure to the principal buyers of the goods.

Since the exposure in any magazine is the number of advertisements multiplied by the number of principal buyers, therefore, the value of \( Z \) is given by:

\[
Z = (0.20 \times 1,00,000) x_1 + (0.15 \times 60,000) x_2 + (0.08 \times 40,000) x_3 \\
= 20,000 x_1 + 9,000 x_2 + 3,200 x_3
\]

The owner of Fancy Goods shop wishes to maximise the total exposure to principal buyers, hence the objective function is given by

Maximize \( Z = 20,000 x_1 + 9,000 x_2 + 3,200 x_3 \)

subject to the following constraints:

(i) The budgeted amount for the advertisement is at the most \( \text{₹} 1,00,000 \).

Hence, \( 8,000 x_1 + 6,000 x_2 + 5,000 x_3 \leq 1,00,000 \).

(ii) Also, the magazine A should have no more than 15 advertisements, B and C each should get at least 8 advertisements.

Hence, \( x_1 \leq 15 \)

and \( x_2 \geq 8, \ x_3 \geq 8 \)

Thus, the linear programming model for the problem is

Maximise \( Z = 20,000 x_1 + 9,000 x_2 + 3,200 x_3 \)

subject to

\[
8,000 x_1 + 6,000 x_2 + 5,000 x_3 \leq 1,00,000 \\
x_1 \leq 15, \ x_2 \geq 8, \ x_3 \geq 8 \]

Where \( x_1, x_2, \) and \( x_3 \geq 0 \)

Illustration 16

A Mutual Fund Company has \( \text{₹} 20 \) lakhs available for investment in Government Bonds, blue chip stocks, speculative stocks and short-term deposits. The annual expected return and risk factor are given below:

<table>
<thead>
<tr>
<th>Type of investment</th>
<th>Annual Expected Return (%)</th>
<th>Risk Factor (0 to 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Government Bonds</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>Blue Chip Stocks</td>
<td>19</td>
<td>24</td>
</tr>
<tr>
<td>Speculative Stocks</td>
<td>23</td>
<td>48</td>
</tr>
<tr>
<td>Short-term Deposits</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>
Mutual fund is required to keep at least ₹ 2 lakhs in short-term deposits and not to exceed average risk factor of 42. Speculative stocks must be at most 20 percent of the total amount invested. How should mutual fund invest the funds so as to maximize its total expected annual return? Formulate this as a Linear Programming Problem. Do not solve it.

Solution

Let \( x_1, x_2, x_3 \) and \( x_4 \) denote the amount of funds to be invested in government bonds, blue chip stocks, speculative stocks and short-term deposits respectively. Let \( Z \) denote the total expected return.

Since the Mutual Fund Company has ₹ 20 lakhs available for investment,
\[
\sum x_1 + x_2 + x_3 + x_4 \leq 20,00,000
\]  
...(i)

Also, Mutual fund is required to keep at least ₹ 2 lakhs in short-term deposits,
Hence, \( x_4 \geq 2,00,000 \)
...(ii)

The average risk factor is given by
\[
\frac{12x_1 + 24x_2 + 48x_3 + 6x_4}{x_1 + x_2 + x_3 + x_4} \leq 42
\]

or \( 12x_1 + 24x_2 + 48x_3 + 6x_4 \leq 42 \left(x_1 + x_2 + x_3 + x_4\right) \)

or \(-30x_1 - 18x_2 + 6x_3 - 36x_4 \leq 0 \)
...(iii)

Further, speculative stock must be at most 20 percent of the total amount invested, hence
\[
x_3 \leq 0.20 \left(x_1 + x_2 + x_3 + x_4\right)
\]

or \(-0.2x_1 - 0.2x_2 + 0.8x_3 - 0.2x_4 \leq 0 \)
...(iv)

Finally, the objective is to maximise the total expected annual return, the objective function for Mutual Fund can be expressed as
Maximise \( Z = 0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4 \)
...(v)

Summarising equations (i) to (v), the linear programming model for the Mutual Fund company is formulated as below:

Objective function :
Maximize \( Z = 0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4 \)
Subject to the constraints

\[ x_1 + x_2 + x_3 + x_4 \leq 20,000 \]
\[ x_4 \geq 2,000 \]
\[ -30x_1 - 18x_2 + 6x_3 - 36x_4 \leq 0 \]
\[ -0.2x_1 - 0.2x_2 + 0.8x_3 - 0.2x_4 \leq 0 \]
where \( x_1 > 0, x_2 > 0, x_3 > 0 \) and \( x_4 > 0 \)

Illustration 17

A firm produces three products A, B and C. It uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below:

<table>
<thead>
<tr>
<th>Raw Material</th>
<th>Requirements per unit of Product</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>I</td>
<td>3</td>
</tr>
<tr>
<td>II</td>
<td>5</td>
</tr>
</tbody>
</table>

The labour time for each unit of product A is twice that of product B and three times that of product C. The entire labour force of the firm can produce the equivalent of 3,000 units. The (marks) minimum demand of the three products is 600, 650 and 500 units respectively. Also, the ratios of the number of units produced must be equal to 2 : 3 : 4. Assuming the profits per unit of A, B and C as \( \₹50, \₹50 \) and \( \₹80 \) respectively.

Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize the profit.

Solution

Let the firm produce \( x_1 \) units of product A, \( x_2 \) units of products B and \( x_3 \) units of product C.

The profit per unit of products A, B and C is \( \₹50 \), and \( \₹80 \) respectively. Since the objective of the firm is to maximize the profit, therefore, the objective function is given by

\[ \text{Maximise } Z = 50x_1 + 50x_2 + 80x_3 \]

The firm uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. As per the given data, the raw material constraints can be formulated as given below:

\[ 3x_1 + 4x_2 + 5x_3 \leq 5,000 \quad \text{...}(i) \]
and \[ 5x_1 + 3x_2 + 5x_3 \leq 7,500 \quad \text{...}(ii) \]

The labour time for each unit of product A is twice that of product B and three times that of product C. Also the entire labour force can produce the equivalent of 3000 units.
\[
\therefore x_1 + \frac{x_2}{2} + \frac{x_3}{3} \leq 3,000
\]

Or

\[6x_1 + 3x_2 + 2x_3 \leq 18000 \quad \text{...(iii)}\]

The minimum demand of the three products is 600, 650 and 500 units respectively.

Hence, \(x_1 \geq 600, \ x_2 \geq 650 \) and \(x_3 \geq 500 \) \quad \text{...(iv)}

Since the ratios of the number of units produced must be equal to 2 : 3 : 4, therefore,

\[\frac{1}{2}x_1 = \frac{1}{3}x_2, \text{ and } \frac{1}{3}x_2 = \frac{1}{4}x_3\]

or \(3x_1 = 2x_2 \) and \(4x_2 = 3x_3\) \quad \text{...(v)}

The linear programming model can be formulated as follows:

Maximise \(Z = 50x_1 + 50x_2 + 80x_3\)

Subject to the constraints:

\[
3x_1 + 4x_2 + 5x_3 \leq 5,000
\]

\[
5x_1 + 3x_2 + 5x_3 \leq 7,500
\]

\[
6x_1 + 3x_2 + 2x_3 \leq 18,000
\]

\[
3x_1 = 2x_2 \text{ and } 4x_2 = 3x_3
\]

\(x_1 \geq 600, \ x_2 \geq 650 \) and \(x_3 \geq 500\).

**Illustration 18**

The Delhi Florist Company is planning to make up floral arrangements for the upcoming festival. The company has available the following supply of flowers at the costs shown:

<table>
<thead>
<tr>
<th>Type</th>
<th>Number available</th>
<th>Cost per flower</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red roses</td>
<td>800</td>
<td>₹ 0.20</td>
</tr>
<tr>
<td>Gardenias</td>
<td>456</td>
<td>₹ 0.25</td>
</tr>
<tr>
<td>Carnations</td>
<td>4,000</td>
<td>₹ 0.15</td>
</tr>
<tr>
<td>White roses</td>
<td>920</td>
<td>₹ 0.20</td>
</tr>
<tr>
<td>Yellow roses</td>
<td>422</td>
<td>₹ 0.22</td>
</tr>
</tbody>
</table>

These flowers can be used in any of the four popular arrangements whose makeup and selling prices are as follows:

<table>
<thead>
<tr>
<th>Arrangement</th>
<th>Requirements</th>
<th>Selling price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economy</td>
<td>4 red roses, 2 gardenias, 8 carnations</td>
<td>₹ 6</td>
</tr>
</tbody>
</table>
May time 8 white roses ₹8
5 gardenias
10 carnations
4 yellow roses
Spring colour 9 red roses ₹10
10 carnations
9 white roses
6 yellow roses
Deluxe rose 12 red roses ₹12
12 white roses
12 yellow roses

Formulate a linear programming problem which allows the florist company to determine how many units of each arrangement should be made up in order to maximize profits assuming all arrangements can be sold.

The formulated LP problem is not required to be solved.

Solution

The profits for each arrangement are:

\[
\begin{align*}
\text{Economy} & : 6.00 - 4(0.20) - 2(0.25) - 8(0.15) = ₹3.50 \\
\text{May time} & : 8.00 - 8(0.20) - 5(0.25) - 10(0.15) - 4(0.22) = ₹2.77 \\
\text{Spring colour} & : 10.00 - 9(0.20) - 10(0.15) - 9(0.20) - 6(0.22) = ₹3.58 \\
\text{Deluxe rose} & : 12.00 - 12(0.20) - 12(0.20) - 12(0.22) = ₹4.56
\end{align*}
\]

Let \( x_1, x_2, x_3, x_4 \) be number of units arrangements of type Economy, May time, Spring colour & Deluxe rose.

Then the objective is \( \text{Maximise } Z = 3.5x_1 + 2.77x_2 + 3.58x_3 + 4.56x_4 \)

subject to

\[
\begin{align*}
4x_1 + 9x_3 + 12x_4 & \leq 800 \\
2x_1 + 5x_2 & \leq 456 \\
8x_1 + 10x_2 + 10x_3 & \leq 4000 \\
8x_2 + 9x_3 + 12x_4 & \leq 920 \\
4x_2 + 6x_3 + 12x_4 & \leq 422 \\
\text{All } x_i's & \geq 0.
\end{align*}
\]
Graphical Method

Illustration 19

A company produces two types of presentation goods A and B that require gold and silver. Each unit of type A requires 3 gms. of silver and 1 gm. of gold while that of B requires 1 gm. of silver and 2 gms. of gold. The company can procure 9 gms of silver and 8 gms. of gold. If each unit of type A brings a profit of ₹ 40 and that of types B ₹ 50, determine the number of units of each type that the company should produce to maximize the profit. What is the maximum profit?

Solution

Let $x_1$ be the number of units of type A of presentation goods to be produced and $x_2$ is that of type B. The formulation of L.P.P. based on the given data may be stated as follows:

Maximize $Z = 40x_1 + 50x_2$

Subject to the constraints

\[ 3x_1 + x_2 \leq 9 \]
\[ x_1 + 2x_2 \leq 8 \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \]

For the line $3x_1 + x_2 = 9$

Let $x_1 = 0$, so that $x_2 = 9$
Let $x_2 = 0$, so that $x_1 = 3$

For the line $x_1 + 2x_2 = 8$

Let $x_1 = 0$, so that $x_2 = 4$
Let $x_2 = 0$, so that $x_1 = 8$
The shaded portion in the diagram represent the feasible region, and the matrix of the extreme points E_i, i = 1, 2, 3, 4 is
\[
E = \begin{bmatrix}
0 & 0 & E_1 \\
0 & 4 & E_2 \\
2 & 3 & E_3 \\
3 & 0 & E_4
\end{bmatrix}
\]

The column vector for the objective function is \( C = \begin{bmatrix} 40 \\ 50 \end{bmatrix} \)

The column vector for the values of the objective function is given by
\[
EC = \begin{bmatrix}
0 & 0 & 0 \\
0 & 4 & 0 \\
2 & 3 & 2 \times 40 + 3 \times 50 \\
3 & 0 & 3 \times 40 + 0 \times 50
\end{bmatrix} = \begin{bmatrix} 0 \\ 200 \\ 230 \\ 120 \end{bmatrix} \]

Since 230 is the maximum value in EC, this value is reached at the extreme point E_3 whose coordinates are (2,3).

Thus, for maximising profit, the company should produce 2 units of type A and 3 units of type B presentation goods.

The maximum profit is ₹ 230.

**Illustration 20**

A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5,760 to invest and has space utmost for 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. His expectation is that he can sell a fan at a profit of ₹ 22 and a sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Formulate this problem as a linear programming problem and then use graphical method to solve it.

**Solution**

Let \( x_1 \) be the number of fans and \( x_2 \) be the number of sewing machines purchased by this dealer.

The formulation of the problem as per given information is as under.

Maximize \( Z = 22 x_1 + 18 x_2 \)

Subject to the constraints
\[ x_1 + x_2 \leq 20 \]
\[ 360x_1 + 240x_2 \leq 5,760 \]
\[ x_1 \geq 0, x_2 \geq 0 \]

For the line \( x_1 + x_2 = 20 \)

let \( x_1 = 0 \), so that \( x_2 = 20 \)

let \( x_2 = 0 \), so that \( x_1 = 20 \)

For the line \( 360x_1 + 240x_2 = 5,760 \)

i.e. \( 3x_1 + 2x_2 = 48 \)

let \( x_1 = 0 \), so that \( x_2 = 24 \)

let \( x_2 = 0 \), so that \( x_1 = 16 \)

The shaded portion in the diagram is the feasible region, and the matrix \( E \) of the extreme points, \( E_i, i = 1, 2, 3, 4 \) is given by

\[
E = \begin{bmatrix}
0 & 0 & E_1 \\
0 & 20 & E_2 \\
8 & 12 & E_3 \\
16 & 0 & E_4
\end{bmatrix}
\]

The column vector for the objective function is

\[
C = \begin{bmatrix} 22 & x_1 \\ 18 & x_2 \end{bmatrix}
\]

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The column vector for the values of the objective function is given by
\[
\begin{bmatrix}
0 & 0 \\
0 & 20 \\
8 & 12 \\
16 & 0 \\
0 & 22 \\
0 & 20 \\
0 & 18 \\
0 & 12 \\
0 & 18 \\
0 & 12 \\
0 & 18 \\
360 & E_1 \\
392 & E_3 \\
352 & E_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \times 22 + 0 \times 18 \\
0 \times 22 + 20 \times 18 \\
8 \times 22 + 12 \times 18 \\
16 \times 22 + 0 \times 18 \\
\end{bmatrix} = \begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
\end{bmatrix}
\]

Since 392 is the largest element of matrix EC, therefore the maximum profit is reached at the extreme point E_3 whose coordinates are (8,12). Thus, to maximize his profit the dealer should buy 8 fans and 12 sewing machines.

**Illustration 21**

A company produces two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table:

<table>
<thead>
<tr>
<th>Process</th>
<th>Product X (hr/unit)</th>
<th>Product Y (hr/unit)</th>
<th>Available capacity per day (hr.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Process I</td>
<td>12</td>
<td>12</td>
<td>840</td>
</tr>
<tr>
<td>Process II</td>
<td>3</td>
<td>6</td>
<td>300</td>
</tr>
<tr>
<td>Process III</td>
<td>8</td>
<td>4</td>
<td>480</td>
</tr>
<tr>
<td>Process per unit (₹)</td>
<td>5</td>
<td>7</td>
<td></td>
</tr>
</tbody>
</table>

How many units of each product should the company produce per day in order to maximize profit?

**Solution**

Let x be the number of units of type X and y be the number of units of type Y products to be produced. The formulation of the problem based on the given data is as follows:

Maximize \( Z = 5x + 7y \)
Subject to the constraints
\( 12x + 12y \leq 840 \)
\( 3x + 6y \leq 300 \)
\( 8x + 4y \leq 480 \)
\( x \geq 0, y \geq 0 \)
The line $12x + 12y = 840$ is equivalent to
\[ x + y = 70 \]
\[
\begin{array}{c|c|c}
    & x & 40 \\ \hline
    y & 30 & 20 \\
\end{array}
\]

The line $3x + 6y = 300$ is equivalent to
\[ x + 2y = 100 \]
\[
\begin{array}{c|c|c}
    & x & 0 \\ \hline
    y & 50 & 25 \\
\end{array}
\]

The line $8x + 4y = 480$ is equivalent to
\[ 2x + y = 120 \]
\[
\begin{array}{c|c|c}
    & x & 60 \\ \hline
    y & 0 & 40 \\
\end{array}
\]

The shaded portion in the diagram (on previous page) represents the feasible region, and the matrix of the extreme points
\[ E_{i} = 1, 2, 3, 4, 5 \]

The column vector for the values of the objective function is given by
\[
\begin{bmatrix}
0 \\
0 \\
50 \\
40 \\
50 \\
60
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
5 \\
7 \\
0 \\
5 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
350 \\
410 \\
390 \\
300
\end{bmatrix}
= E_1, E_2, E_3, E_4, E_5
\]

Since 410 is the largest element in EC, the maximum value is achieved at the extreme point $E_3$ whose coordinates are (40, 30).

Thus to maximize profit, the company should produce 40 units of X and 30 units of Y.

**Illustration 22**

A company that produces soft drinks has a contract that requires that a minimum of 80 units of the chemical A and 60 units of the chemical B into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X1 has a mix of 4 units of A and 2 units of B that costs ₹10, and supplier X2 has a mix of 1 unit of A and 1 unit of B that costs ₹4. How many mixes from company X1 and company X2 should the company purchase to honour contract requirement and yet minimize cost?
Solution
The problem may be summarized as follows:

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Chemical A</th>
<th>Chemical B</th>
<th>Cost per mix (₹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supplier X₁</td>
<td>4</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>Supplier X₂</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Let \( x_1 \) be the number of mixes to be purchased from supplier \( X_1 \) and \( x_2 \) be of those to be purchased from supplier \( X_2 \).

The conditions of the problem when symbolised, take the form:

Minimize \( Z = 10x_1 + 4x_2 \)

Subject to the restrictions

\[
4x_1 + x_2 \geq 80
\]

\[
2x_1 + x_2 \geq 60
\]

\[
x_1 \geq 0, \ x_2 \geq 0
\]

For the line \( 4x_1 + x_2 = 80 \)

let \( x_1 = 0 \), so that \( x_2 = 80 \)

let \( x_2 = 0 \), so that \( x_1 = 20 \)

For the line \( 2x_1 + x_2 = 60 \)

let \( x_1 = 0 \), so that \( x_2 = 60 \)

let \( x_2 = 0 \), so that \( x_1 = 30 \)
Feasible region is shaded in the diagram which appears to be unbounded. We now try to determine the additional hidden conditions in the problem for which the feasible region becomes bounded.

Each bottle of the drink contains 80 units of the chemical A and 60 units of the chemical B. If the drinks are made from the mixes of company X₁ only, each bottle would require 20 mixes to meet the chemical A requirement and 30 mixes to meet the chemical B requirement. Therefore, 30 mixes would meet all minimum requirements. Any lesser amount of mixes would not meet the minimum requirement for chemical B.

Similarly, if the drinks are made from the mixes of company X₂ only, each bottle would require 80 mixes to meet both the chemical requirements.

Since the company that manufactures drinks wants to make a blending of the mixes purchased from both the companies X₁ and X₂, it is possible that the company would need less than the maximum amount of both supplies to meet the guaranteed minimum. Thus we add the two additional hidden conditions $x₁ < 30$, $x₂ < 80$.

With these two additional restrictions the above diagram takes the form:

```
\begin{array}{cc}
x₁ & x₂ \\
0 & 80 \\
10 & 40 \\
30 & 0 \\
30 & 80 \\
\end{array}
```

The matrix of the extreme points, in this case, is $E = \begin{bmatrix} E₁ \\ E₂ \\ E₃ \\ E₄ \end{bmatrix}$

The column vector for the objective function is $C = \begin{bmatrix} 10 \\ 4 \end{bmatrix} x₁ \begin{bmatrix} x₂ \end{bmatrix}$
10.54 Advanced Management Accounting

The column vector for the values of the objective function is given by

\[
\begin{bmatrix}
0 & 80 \\
10 & 40 \\
30 & 0 \\
30 & 80
\end{bmatrix}
\begin{bmatrix}
0 \\
10 \\
30 \\
80
\end{bmatrix}
=\begin{bmatrix}
0 \times 10 + 80 \times 4 \\
10 \times 10 + 40 \times 4 \\
30 \times 10 + 0 \times 4 \\
30 \times 10 + 80 \times 4
\end{bmatrix}
=\begin{bmatrix}
320 \\
260 \\
300 \\
620
\end{bmatrix}
\]

Since 260 is the smallest element in EC, the minimum value is reached at the extreme point \(E_2\), whose coordinates are (10,40).

Thus, to honour the contract and yet to minimize cost, the company should purchase 10 mixes from \(X_1\) and 40 mixes from \(X_2\).

**Illustration 23**

A local travel agent is planning a charter trip to a major sea resort. The eight day/seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accommodation, meal plans and tour options. The following table summarises the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an aircraft for the flat fee of \(₹2,00,000\) for the entire trip.

<table>
<thead>
<tr>
<th>Tour Plan</th>
<th>Price (₹)</th>
<th>Hotel Costs (₹)</th>
<th>Meals &amp; other Expenses (₹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deluxe</td>
<td>10,000</td>
<td>3,000</td>
<td>4,750</td>
</tr>
<tr>
<td>Standard</td>
<td>7,000</td>
<td>2,200</td>
<td>2,500</td>
</tr>
<tr>
<td>Economy</td>
<td>6,500</td>
<td>1,900</td>
<td>2,200</td>
</tr>
</tbody>
</table>

In planning the trip, the following considerations must be taken into account:

(i) At least 10 percent of the packages must be of the deluxe type.
(ii) At least 35 percent but not more than 70 percent must be of the standard type.
(iii) At least 30 percent must be of the economy type.
(iv) The maximum number of deluxe packages available in any aircraft is restricted to 60.
(v) The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the number of packages to offer in each type so as to maximize the total profit.

(a) Formulate the above as a linear programming problem.
(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.

(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results.

Solution

Let $x_1$, $x_2$, $x_3$ denote the number of Deluxe, Standard & Economy tour packages to be offered to 200 persons that will maximize the profit. In other words, the total number of tours of three types offered by the concern is restricted to 200 only to maximize its profits.

The contribution (per person) arising out of each type of tour package offered is as follows:

<table>
<thead>
<tr>
<th>Packages offered</th>
<th>Price (₹) (1)</th>
<th>Hotel Costs (₹) (2)</th>
<th>Meals &amp; other Expenses (₹) (3)</th>
<th>Net Profit (₹) (4)= (1)–[(2)+(3)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deluxe</td>
<td>10,000</td>
<td>3,000</td>
<td>4,750</td>
<td>2,250</td>
</tr>
<tr>
<td>Standard</td>
<td>7,000</td>
<td>2,200</td>
<td>2,500</td>
<td>2,300</td>
</tr>
<tr>
<td>Economy</td>
<td>6,500</td>
<td>1,900</td>
<td>2,200</td>
<td>2,400</td>
</tr>
</tbody>
</table>

The travel agent has to pay the flat fee of ₹ 2,00,000 for the chartered aircraft for the entire trip. Consequently the objective function (profit function) will be:

$$\text{Max } Z = ₹ 2,250x_1 + ₹ 2,300x_2 + ₹ 2,400x_3 - ₹ 2,00,000$$

The objective of the profit function is to maximize the profit.

The constraints based on the given conditions (i) to (v) are as follows:

- $x_1 \geq 20$ from condition (i)
- $x_2 \geq 70$ from condition (ii)
- $x_3 \geq 60$ from conditions (iv)
- $x_1 + x_2 \geq 120$ from condition (v)
- Also $x_1 + x_2 + x_3 = 200$, $x_1, x_2, x_3 \geq 0$

On combining suitably, the above constraints reduce to the following constraints:

- $20 \leq x_1 \leq 60$
- $70 \leq x_2 \leq 140$
- $x_3 \geq 60$
- $x_1 + x_2 \geq 120$
- $x_1 + x_2 + x_3 = 200$ & $x_1, x_2, x_3 \geq 0$

(a) The linear programming problem formulated from the given data is (₹):

$$\text{Max } Z = ₹ 2,250x_1 + ₹ 2,300x_2 + ₹ 2,400x_3 - ₹ 2,00,000$$

Subject to constraints

- $20 \leq x_1 \leq 60$
- $70 \leq x_2 \leq 140$
10.56 Advanced Management Accounting

\[ x_3 \geq 60 \\
\geq 120 \\
+ x_3 = 200 \\
, x_1, x_2, x_3 \geq 0 \\
\]

(b) Since \( x_1 + x_2 + x_3 = 200 \) or \( x_3 = 200 - (x_1 + x_2) \)

Substitute the value of \( x_3 \) in the relations mentioned under (a), the linear programming problem reduces to the following:

Maximize \( Z = -150x_1 + 100x_2 + 28,00,000 \)

Subject to \( 20 \leq x_1 \leq 60 \)

\[ 70 \leq x_2 \leq 14 \]

\[ 120 \leq x_1 + x_2 \leq 140 \]

\& \( x_1, x_2 \geq 0 \)

(c) Refer to the figure drawn for the optimum solution of the restated L.P. problem under (b).

<table>
<thead>
<tr>
<th>Point</th>
<th>Co-ordinates of the corner points of the feasible region (values of ( x_1 ) and ( x_2 ))</th>
<th>Values of the objective function ( Z = -150x_1 - 100x_2 + 2,80,000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>((50,70))</td>
<td>(\₹ 2,65,500)</td>
</tr>
<tr>
<td>( B )</td>
<td>((60,70))</td>
<td>(\₹ 2,64,000)</td>
</tr>
<tr>
<td>( C )</td>
<td>((60,80))</td>
<td>(\₹ 2,63,000)</td>
</tr>
<tr>
<td>( D )</td>
<td>((20,120))</td>
<td>(\₹ 2,65,000)</td>
</tr>
<tr>
<td>( E )</td>
<td>((20,100))</td>
<td>(\₹ 2,67,000)</td>
</tr>
</tbody>
</table>
The maximum profit is attained at the corner point E whose co-ordinates are (20,100)

Interpretation: The profit of the objective function stated under (a) becomes maximum i.e. \( \text{₹} \ 2,67,000 \) when 
\[
\begin{align*}
  x_1 &= 20; \\
  x_2 &= 100 \quad \text{and} \quad x_3 = 80 \\
  \therefore x_3 &= 200 - (x_1 + x_2) = 200 - (20 + 100)
\end{align*}
\]

Simplex Method-Maximization

Illustration 24

An electronics firm is undecided as to the most profitable mix for its products. The products now manufactured are transistors, resistors and carbon tubes with a profit (per 100 unit) of \( \text{₹} \ 10, \text{₹} \ 6 \) and \( \text{₹} \ 4 \) respectively. To produce a shipment of transistors containing 100 units requires 1 hour of engineering, 10 hours of direct labours and 2 hours of administrative service. To produce 100 resistors are required 1 hour, 4 hours and 2 hours of engineering, direct labour and administrative time respectively. To produce on shipment of the tubes (100 units) requires 1 hour of engineering, 5 hours of direct labour and 6 hours of administration. There are 100 hours of engineering services available, 600 hours of direct labour and 300 hours of administration. What is the most profitable mix?

Solution

Let us tabulate the data in convenient manner.

<table>
<thead>
<tr>
<th>Products</th>
<th>Transistors</th>
<th>Resistors</th>
<th>Carbon Tubes</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engineering</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>Labour</td>
<td>10</td>
<td>4</td>
<td>5</td>
<td>600</td>
</tr>
<tr>
<td>Administration</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>300</td>
</tr>
<tr>
<td>Profit (per 100 units) (₹ )</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Symbolically, Maximize \( 10x + 6y + 4z \)

here \( x, y, z \) represents units of Transistors, Resistors and Carbon tubes (in 100 units)

Subject to
\[
\begin{align*}
  1x + 1y + 1z &\leq 100 \\
  10x + 4y + 5z &\leq 600 \\
  2x + 2y + 6z &\leq 300 \\
  x \geq 0, y \geq 0, z \geq 0
\end{align*}
\]

Introducing slack variables

Maximize \( Z = 10x + 6y + 4z + 0s_1+0s_2+0s_3 \)

Subject to
\[
\begin{align*}
  x + y + z + s_1 &= 100 \\
  10x + 4y + 5z + s_2 &= 600 \\
  2x + 2y + 6z + s_3 &= 300 \\
  x \geq 0, y \geq 0, z \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0
\end{align*}
\]
### SIMPLEX TABLEAU-I

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables (B)</th>
<th>Cb</th>
<th>Value of Basic Variables (b=XB)</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>s₁</td>
<td>0</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>s₂</td>
<td>0</td>
<td>600</td>
<td>10</td>
<td>4</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>1/5</td>
<td>s₃</td>
<td>0</td>
<td>300</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>150</td>
</tr>
</tbody>
</table>

\[ Z_j = \sum C_{Bj}X_j \]

\[ C_j - Z_j = 10, 6, 4, 0, 0, 0 \]

### SIMPLEX TABLEAU-II

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables (B)</th>
<th>Cb</th>
<th>Value of Basic Variables (b=XB)</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>s₁</td>
<td>0</td>
<td>40</td>
<td>6/10</td>
<td>5/10</td>
<td>1</td>
<td>-1/10</td>
<td>0</td>
<td>400/6=67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>10</td>
<td>60</td>
<td>4/10</td>
<td>5/10</td>
<td>0</td>
<td>1/10</td>
<td>0</td>
<td>600/4=150</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s₃</td>
<td>0</td>
<td>180</td>
<td>12/10</td>
<td>5</td>
<td>0</td>
<td>-2/10</td>
<td>1</td>
<td>150</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ Z_j = \sum C_{Bj}X_j \]

\[ C_j - Z_j = 0, 2, -1, 0, -1, 0 \]

### SIMPLEX TABLEAU-III

<table>
<thead>
<tr>
<th>Fixed Ratio</th>
<th>Basic Variables (B)</th>
<th>Cb</th>
<th>Value of Basic Variables (b=XB)</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
<th>Replacement Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>6</td>
<td>400/6</td>
<td>0</td>
<td>1</td>
<td>5/6</td>
<td>10/6</td>
<td>-1/6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>10</td>
<td>100/3</td>
<td>1</td>
<td>0</td>
<td>1/6</td>
<td>-2/3</td>
<td>1/6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>s₃</td>
<td>0</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ Z_j = \sum C_{Bj}X_j \]

\[ C_j - Z_j = 10, 6, 20/3, 10/3, +2/3, 0 \]

\[ C_j - Z_j = 0, 0, -8/3, -10/3, -2/3, 0 \]

\[ C_j - Z_j \text{ Since all entries in index row (C}_j - Z_j) \text{ are –ve or zero. Current solution is optimum.} \]
Hence the most profitable mix is \( y = \frac{400}{6} \) Resistors and \( x = \frac{100}{3} \) Transistors. The maximum profit is \( 400 + \frac{1000}{3} = ₹ \frac{733}{3} \).

**Illustration 25**

For a company engaged in the manufacture of three products viz., X, Y and Z, the available data are given in Tables 1, 2 and 3 below:

**Table 1 : Minimum sales requirements**

<table>
<thead>
<tr>
<th>Product</th>
<th>Minimum Sales Requirements per month</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>10</td>
</tr>
<tr>
<td>Y</td>
<td>20</td>
</tr>
<tr>
<td>Z</td>
<td>30</td>
</tr>
</tbody>
</table>

**Table 2 : Operations, Required Processing Times and Capacity**

<table>
<thead>
<tr>
<th>Operations</th>
<th>Time (Hrs) required per item of</th>
<th>Total available hours per month</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 3 : Profit (₹) per unit**

<table>
<thead>
<tr>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
</tr>
<tr>
<td>Y</td>
</tr>
<tr>
<td>Z</td>
</tr>
</tbody>
</table>

Find out the product mix to maximize profit.

**Solution**

Let \( x, y \) and \( z \) denote the number of units produced per month for the products X, Y and Z respectively.

Minimum sales requirements give the following constraints:

\[
\begin{align*}
x & \geq 10 \\
y & \geq 20 \\
z & \geq 30 \\
\end{align*}
\]

where \( x, y, z \geq 0 \)

Operations, processing times and capacity lead to the following constraints:

\[
x + 2y + 2z \leq 200
\]
The objective function is
Maximize $10x + 15y + 8z$
Thus, we have to solve the following problem
Maximize $P = 10x + 15y + 8z$
Subject to
\[ \begin{align*}
  x + 2y + 2z & \leq 200 \quad \text{(i)} \\
  2x + y + z & \leq 220 \quad \text{(ii)} \\
  3x + y + 2z & \leq 180 \quad \text{(iii)} \\
  x & \geq 10 \quad \text{(iv)} \\
  y & \geq 20 \quad \text{(v)} \\
  z & \geq 30 \quad \text{(vi)} \\
  x, y, z & > 0
\end{align*} \]
Let us make following substitutions
\[\begin{align*}
  x &= a + 10 \\
  y &= b + 20 \\
  z &= c + 30
\end{align*}\]
Where $a, b, c \geq 0$
Substituting these values in the objective function and constraints (i), (ii), & (iii), the problem becomes
Maximize $P = 10a + 15b + 8c + 640$
Subject to
\[\begin{align*}
  (a + 10) + 2(b + 20) + 2(c + 30) & \leq 200 \\
  2(a + 10) + (b + 20) + (c + 30) & \leq 220 \\
  3(a + 10) + (b + 20) + 2(c + 30) & \leq 180
\end{align*}\]
Where, $a, b, c \geq 0$
Or, Maximize $P = 10a + 15b + 8c + 640$
Subject to
\[\begin{align*}
  a + 2b + 2c & \leq 90 \\
  2a + b + c & \leq 150 \\
  3a + b + 2c & \leq 70
\end{align*}\]
Where, $a, b, c \geq 0$
Adding slack variables $s_1, s_2,$ and $s_3$, we get
Maximize $P = 10a + 15b + 8c + 640 + 0s_1 + 0s_2 + 0s_3$
Subject to

\[
\begin{align*}
 a + 2b + 2c + s_1 &= 90 \\
 2a + b + c + s_2 &= 150 \\
 3a + b + 2c + s_3 &= 70
\end{align*}
\]

The initial feasible solution is obtained by setting \( a = b = c = 0 \). The solution is \( s_1 = 90 \), \( s_2 = 150 \), \( s_3 = 70 \), \( P = 640 \). This solution and further improved solutions are presented in the following matrix form:

**SIMPLEX TABLEAU-I**

<table>
<thead>
<tr>
<th>( C_B )</th>
<th>Basic Variable ( (B) )</th>
<th>Value of Basic Variables ( (=X_B) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( s_1 )</td>
<td>90</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( 45 )</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>150</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( 150 )</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>70</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( 70 )</td>
</tr>
</tbody>
</table>

\[ Z_i = \sum C_{Bi}X_i \]

\[ C_j - Z_i = 10 \quad 15 \quad 8 \quad 0 \quad 0 \quad 0 \]

**SIMPLEX TABLEAU-II**

<table>
<thead>
<tr>
<th>( C_B )</th>
<th>Basic Variable ( (B) )</th>
<th>Value of Basic Variables ( (=X_B) )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>( b )</td>
<td>45</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>0</td>
<td>( 90 )</td>
</tr>
<tr>
<td>0</td>
<td>( s_2 )</td>
<td>105</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td>0</td>
<td>( 70 )</td>
</tr>
<tr>
<td>0</td>
<td>( s_3 )</td>
<td>25</td>
<td>5/2</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>1</td>
<td>( 10 )</td>
</tr>
</tbody>
</table>

\[ Z_i = \sum C_{Bi}X_i \]

\[ C_j - Z_i = 15/2 \quad 15 \quad 15 \quad 15/2 \quad 0 \quad 0 \]

\[ C_j - Z_i = 5/2 \quad 0 \quad -7 \quad -15/2 \quad 0 \quad 0 \]
SIMPLEX TABLEAU-III

<table>
<thead>
<tr>
<th>C_i</th>
<th>10</th>
<th>15</th>
<th>8</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_b</td>
<td>Basic Variable (B)</td>
<td>Value of Basic Variables b(=X_b)</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>s_1</td>
<td>s_2</td>
</tr>
<tr>
<td>15</td>
<td>b</td>
<td>40</td>
<td>0</td>
<td>1</td>
<td>4/5</td>
<td>3/5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>s_2</td>
<td>90</td>
<td>0</td>
<td>0</td>
<td>-3/5</td>
<td>-1/5</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>a</td>
<td>10</td>
<td>1</td>
<td>0</td>
<td>2/5</td>
<td>-7/5</td>
<td>0</td>
</tr>
</tbody>
</table>

Z_i = \sum C_{b_i}X_{i} = 10 15 16 7 0 1

C_i - Z_i = 0 0 -8 -7 0 -1

Since all NER elements are less than or equal to zero, the previous table gives the optimal solution. The solution is

a = 10, b = 40 and c = 0

Substituting these values

x = 10 + 10 = 20,
y = 40 + 20 = 60,
z = 0 + 30 = 30

and the value of objective function is given by

P = ₹ 10 × 20 + ₹ 15 × 60 + ₹ 8 × 30 = ₹ 200 + ₹ 900 + ₹ 240 = ₹ 1,340

The optimal product mix is to produce 20 units of X, 60 units of Y and 30 units of Z to get a profit of ₹ 1,340.

Simplex Method-Minimization

Illustration 26

You are required to solve following linear programming problem.

Minimize Z=30 x_1+40 x_2

Subject to the constraints

10 x_1+15 x_2≥450
20 x_1+15 x_2≥600
x_1, x_2≥0

Solution

Expressing the problem in the standard form by adding surplus variables s_1 and s_2 and artificial variables A_1 and A_2, M is a large number.

Minimize Z=30 x_1+40 x_2+0s_1 + 0s_2 +M A_1+M A_2
Subject to the constraints
10 \(x_1\) + 15 \(x_2\) - \(s_1\) + \(A_1\) = 450
20 \(x_1\) + 15 \(x_2\) - \(s_2\) + \(A_2\) = 600
\(x_1\), \(x_2\), \(s_1\), \(s_2\), \(A_1\), \(A_2\) \(\geq 0\)

**SIMPLEX TABLEAU-I**

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>Basic Variable (B)</th>
<th>Value of Basic Variables (b(=X_b))</th>
<th>30</th>
<th>40</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>(A_1)</td>
<td></td>
<td>10</td>
<td>15</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>45</td>
</tr>
<tr>
<td>M</td>
<td>(A_2)</td>
<td></td>
<td>20</td>
<td>15</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>30</td>
</tr>
<tr>
<td>Zj</td>
<td></td>
<td></td>
<td>30M</td>
<td>30M</td>
<td>-M</td>
<td>-M</td>
<td>M</td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>Cj-Zj</td>
<td></td>
<td></td>
<td>30-30M</td>
<td>40-30M</td>
<td>M</td>
<td>M</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

30-30M is largest –ve

**SIMPLEX TABLEAU-II**

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>Basic Variable (B)</th>
<th>Value of Basic Variables (b(=X_b))</th>
<th>30</th>
<th>40</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>(A_1)</td>
<td></td>
<td>0</td>
<td>15/2</td>
<td>-1</td>
<td>1/2</td>
<td>1</td>
<td>-1/2</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>(x_1)</td>
<td></td>
<td>1</td>
<td>3/4</td>
<td>0</td>
<td>-1/20</td>
<td>0</td>
<td>1/20</td>
<td>40</td>
</tr>
<tr>
<td>Zj</td>
<td></td>
<td></td>
<td>30</td>
<td>15/2M+45/2</td>
<td>-M</td>
<td>M/2-3/2</td>
<td>M</td>
<td>M/2+3/2</td>
<td></td>
</tr>
<tr>
<td>Cj-Zj</td>
<td></td>
<td></td>
<td>0</td>
<td>15/2M+35/2</td>
<td>M</td>
<td>-M/2+3/2</td>
<td>0</td>
<td>3/2M-3/2</td>
<td></td>
</tr>
</tbody>
</table>

-15/2M+35/2 is largest –ve

**SIMPLEX TABLEAU-III**

<table>
<thead>
<tr>
<th>(C_B)</th>
<th>Basic Variable (B)</th>
<th>Value of Basic Variables (b(=X_b))</th>
<th>30</th>
<th>40</th>
<th>0</th>
<th>0</th>
<th>M</th>
<th>M</th>
<th>Minimum Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>(x_2)</td>
<td></td>
<td>0</td>
<td>1</td>
<td>-2/15</td>
<td>1/15</td>
<td>2/15</td>
<td>-1/15</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>(x_1)</td>
<td></td>
<td>1</td>
<td>0</td>
<td>1/10</td>
<td>-1/10</td>
<td>-1/10</td>
<td>1/10</td>
<td></td>
</tr>
<tr>
<td>Zj</td>
<td></td>
<td></td>
<td>30</td>
<td>40</td>
<td>-7/3</td>
<td>-1/3</td>
<td>7/3</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>Cj-Zj</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
<td>7/3</td>
<td>1/3</td>
<td>M-7/3</td>
<td>M-1/3</td>
<td></td>
</tr>
</tbody>
</table>
Since all entries in Cj-Zj row are either zero or positive in table III, the current solution is the optimum one and is given by:

\[ x_1 = 15, \quad x_2 = 20, \quad \text{Minimum } Z = 1,250 \]

### 10.10 Limitations of Linear Programming

Important limitations of linear programming problems are as follows:

(i) A primary requirement of linear programming is that the objective function and every constraint function must be linear. This requires that the measure of effectiveness and resource usage must be proportional to the level of each activity conducted individually. However, programming problems which are non-linear arise quite frequently. It is occasionally possible to reformulate a non-linear programming problem into the linear programming format so that the simplex method can be used. This is, however, the fortunate exception rather than the rule.

(ii) It may not be possible to solve those problems using linear programming, in which non-linearity arises because of joint interactions between some of the activities regarding the total measure of effectiveness or total usage of some resource. Hence linear programming problem requires that the total measure of effectiveness and total resource usage resulting from the joint performance of the activities must equal the respective sums of these quantities resulting from each activity being performed individually.

In some situations, it may not be true. For example, consider a situation where by a product is produced with the scrap material from the primary product. The material would still have to be procured if only one of the two products were produced. However, the total material requirements if both products are produced is less than the sum of requirements if each were produced individually. It may not be possible to handle such situation with linear programming problems.

(iii) In linear programming problem, fractional values are permitted for the decisions variables. However, many decision problems require that the solution for decision variable should be obtained in non-fractional values. Rounding-off the values obtained by linear programming techniques may not result into an optimal solution in such cases.

(iv) In linear programming problem, coefficients in the objective function and the constraint equations must be completely known and they should not change during the period of study i.e. they should be known constraints. In practical situation, it may not be possible to state all coefficients in the objective function and constraints with certainty. Furthermore, these coefficients may actually be random variables, each with an underlying probability distribution for the values. Such problems cannot be solved using linear programming.